ON NEEDED REALS

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Abstract. Following Blass [4], we call a real a "needed" for a binary relation R on the reals if in every R-adequate set we find an element from which a is Turing computable. We show that every real needed for $\mathbf{Cof}(\mathcal{N})$ is hyperarithmetic. Replacing "R-adequate" by "R-adequate with minimal cardinality" we get related notion of being "weakly needed". We show that is is consistent that the two notions do not coincide for the reaping relation. (They coincide in many models.) We show that not all hyperarithmetical reals are needed for the reaping relation. This answers some questions asked by Blass at the Oberwolfach conference in December 1999 and in [4].

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0. Introduction

We consider some aspects of the following notions:

Definition 0.1. *(1) (Needed reals). Suppose that we have a cardinal characteristic $\mathfrak{x}$ of the reals of the following form: There are (in most cases: Borel) sets $A_-, A_+ \subseteq \mathbb{R}$ and there is a (in most cases: Borel) relation*

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$R \subseteq A_- \times A_+$ such that

$$\mathfrak{r} = ||R|| := \min\{|Y| \,:\, Y \subseteq A_+ \,\wedge\, (\forall x \in A_-)(\exists y \in Y)R(x,y)\}.$$

We call $||R||$ the norm of R. A set $Y \subseteq A_+$ is called R-adequate if $(\forall x \in \mathrm{dom}(R))\ (\exists y \in Y)xRy$. We say that $\eta \in {}^\omega 2$ is needed for R if for every R-adequate set Y there is some $y \in Y$ such that $\eta \leq_T y$.

If $A_+ \not\subseteq \mathbb{R}$ but can be mapped continuously and injective and computably into $\mathbb{R}$ by a mapping c, called a coding, then we call the real a needed for R and c if for any R-adequate set $Y \subseteq A_+$ there is some $y \in Y$ such that $a \leq_T c(y)$. In this situation, a real a is called needed for R, if it is needed for R and c for any coding c.

(2) *(Weakly needed Reals). We call a real a weakly needed for R if for any R-adequate set Y* of minimal cardinality *there is some $y \in Y$ such that $a \leq_T y$.*

Every needed real is weakly needed. Sections 3 to 6 will give some information on the reverse direction.

1. Needed reals for $\mathbf{Cof}(\mathcal{N})$

In this section we answer Blass' question whether only hyperarithmetic reals are needed for the cofinality relation on the ideal of Lebesgue null sets affirmatively.

In this section we work with two particular relations on the reals: For functions $f, g \colon \omega \to \omega$ we write $f \leq^* g$ and say g eventually dominates f if $(\exists n < \omega)(\forall k \geq n)(f(k) \leq g(k))$. The dominating relation is

$$\mathbf{D} = \{(f,g) \,:\, f,g \in {}^\omega\omega \,\wedge\, f \leq^* g\},$$

and the cofinality relation for the ideal of sets of Lebesgue measure zero is

$$\mathbf{Cof}(\mathcal{N}) = \{(F,G) \,:\, F,G \text{ are } F_\sigma\text{-sets of Lebesgue measure 0 and } F \subseteq G\}.$$

We write $\mathbf{cof}(\mathcal{N})$ for $||\mathbf{Cof}(\mathcal{N})||$.

Before stating our first theorem, we review some notation: For $s \in {}^{\omega >}2 = \{t \,:\, (\exists m \in \omega)t \colon m \to 2\}$, we write $\lg(s) = \mathrm{dom}(s)$. If $s \in {}^{\omega>}2$ and $t \in {}^{\omega\geq}2$, we write $s \trianglelefteq t$ if $s = t \restriction \lg(s)$. Let $s \triangleleft t$ denote that $s \trianglelefteq t$ and $s \neq t$. A subset $T \subseteq {}^{\omega>}2$ is called a tree if it is downward closed, i.e., if for all $t \in T$ for all $s \trianglelefteq t$, we have that $s \in T$. We let $\lim(T) = \{f \in {}^\omega 2 \,:\, (\forall n \in \omega) f \restriction n \in T\}$. An element $s \in T$ is a leaf if there is no $t \in T$ such that $s \triangleleft t$. For a tree $T \subseteq {}^{<\omega}2$ and some $\rho \in {}^{<\omega}2$ we set $T^{[\rho]} = \{s \in T \,:\, s \trianglelefteq \rho \vee \rho \trianglelefteq s\}$.

Leb denotes the Lebesgue measure on the measurable subsets of ${}^{\omega}2$, the product space of ω copies of the space $\{0,1\}$, where each point has measure $\frac{1}{2}$.

We deal with the following forcings, where the first is the ordinary Amoeba forcing.

$$\mathbb{Q} = \left\{ T : T \subseteq {}^{\omega>}2, T \text{ is a tree and } \mathrm{Leb}(\lim(T)) > \frac{1}{2} \right\},$$
$$\hat{\mathbb{Q}} = \left\{ T \in \mathbb{Q} : \lim \left\langle \frac{|T \cap {}^{n}2|}{2^n} : n \in \omega \right\rangle > \frac{1}{2} \text{ and } T \text{ has no leaves} \right\},$$

We set $h_T(\rho) = \mathrm{Leb}(\lim(T^{[\rho]}))$.

$$\mathbb{Q}^- = \left\{ T \in \mathbb{Q} : \left(\forall n \in \omega)(\rho \in {}^{n}2 \cap T \to h_T(\rho) \cdot 2^{2^n} \in \omega \setminus \{0\} \right) \right\},$$
$$\hat{\mathbb{Q}}^- = \hat{\mathbb{Q}} \cap \mathbb{Q}^-.$$

The partial order on $\mathbb{Q}$ and its variants is inclusion: subtrees are stronger ($\geq$, we follow the Jerusalem convention) conditions. It is easy to see that $\hat{\mathbb{Q}}$, $\mathbb{Q}^-$ and $\hat{\mathbb{Q}}^-$ are dense suborders of $\mathbb{Q}$.

Theorem 1.1. *Let G be $\hat{\mathbb{Q}}$-generic over V. Then in $V[G]$ the following holds: For every $\eta \in {}^{\omega}2 \cap V$, if η is recursive in the generic tree $T = \bigcap G$, then η is needed for domination.*

Conclusion 1.2. *Since being needed for domination is a an absolute notion (see [6, 9] or 4.1), also in V, every η such that $\eta \in {}^{\omega}2 \cap V$ is recursive in $V[G]$ in the generic tree $T = \bigcap G$, is needed for domination.*

Proof of 1.1. For some $p \in \hat{\mathbb{Q}}$, $\eta \in {}^{\omega}2$, both in V, and Turing machine M (w.l.o.g. also in V) we have that

$$(*) \qquad p \Vdash \text{“}M \text{ computes } \eta \text{ from } \underset{\sim}{T}\text{”}.$$

Let $n(*) \in [1, \omega)$ and $p^* \in \hat{\mathbb{Q}}^-$ be such that $p \leq p^*$ and $\mathrm{Leb}(\lim(p^*)) = \frac{1}{2} + \frac{1}{n(*)}$. Then, by the Lebesgue density theorem (3.10 in [7]), we may choose $m(*)$ such that for any $m \geq m(*)$,

$$\frac{1}{2} + \frac{1}{n(*)} \leq \frac{|p^* \cap ({}^{m}2)|}{2^m} \leq \frac{1}{2} + \frac{1}{n(*)} + \frac{1}{2^{n(*)+7}}.$$

In order to derive from $(*)$ some computation of η relative to a suitable member of a given **D**-adequate set, we shall work with the following trees.

Definition 1.3. *For $r \in \hat{\mathbb{Q}}$ and $\varepsilon > 0$, if* $\mathrm{Leb}(\lim(r)) \geq \frac{1}{2} + \varepsilon$ *let*

$$T^\varepsilon_{r,n} = \Big\{ (q \cap {}^{n>}2, h_q \restriction {}^{n>}2) \, : r \leq q \in \hat{\mathbb{Q}}^-, \\ \mathrm{Leb}(\lim(q)) \geq \frac{1}{2} + \varepsilon, \, \forall m \frac{|q \cap {}^m 2|}{2^m} \geq \frac{1}{2} + \varepsilon \Big\}.$$

We set $T^\varepsilon_r = \bigcup\{T^\varepsilon_{r,n} \, : \, n \in \omega\}$. For $x \in T^\varepsilon_{r,n}$ we write $x = (x(1), x(2))$. We order T^ε_r by $\leq_T$: $(q \cap {}^{n>}2, h_q \restriction {}^{n>}2) \leq_T (q' \cap {}^{n'>}2, h_{q'} \restriction {}^{n'>}2)$ iff $n \leq n'$ and $q \cap {}^{n>}2 = q' \cap {}^{n>}2$ and $h_{q'} \restriction {}^{n>}2 = h_q \restriction {}^{n>}2$. Equivalently, we may consider $t \in T^\varepsilon_{r,n}$ as a function $t \colon q \cap {}^{n>}2 \to \mathbb{R}$, $t(\rho) = h_q(\rho)$. We equip T^ε_r with the tree topology given by $\leq_T$, i.e., basic open sets in the topology are $\{t \in T^\varepsilon_r \, : \, t \geq_T t_0\}$, $t_0 \in T^\varepsilon_r$.

These trees exhibit the following properties:

$(*)_0$ T^ε_r is a tree with finite levels, the nth level being $T^\varepsilon_{r,n}$.

$(*)_1$ If $\langle t_n \, : \, n \in \omega\rangle$ is an ω-branch of T^ε_r then $\mathrm{Leb}(\lim(\bigcup t_n(1)) \geq \frac{1}{2} + \varepsilon$ and if $\varepsilon > 0$ then $\bigcup t_n(1) \in \mathbb{Q}$.

$(*)_2$ Moreover, we have if $r_1 \leq r_2$ in $\hat{\mathbb{Q}}$ and $\mathrm{Leb}(\lim(r_2)) - \frac{1}{2} \geq \varepsilon$, then $T^\varepsilon_{r_2} \subseteq T^\varepsilon_{r_1}$.

$(*)_3$ If $\mathrm{Leb}(\lim(r)) \geq \frac{1}{2} + \varepsilon$, $p^* \leq r \in \mathbb{Q}^-$ and $n \in \omega$ and $\langle t_\ell \, : \, \ell \in \omega\rangle$ is an ω-branch of T^ε_r, then for some $m \in \omega$, there is $t^* \subseteq \mathrm{dom}(t_m)$ (here we regard t's as functions) such that

(a) $\sum\{t_m(\rho) \, : \, \rho \in t^* \cap {}^m 2\} > \frac{1}{2}$.

(b) If M runs with input n and oracle f_{m,t^*} it will give the value $\eta(n)$, where $f_{m,t^*} \colon {}^{m\geq}2 \to \{0,1\}$, $f_{m,t^*}(\rho) = 1 \Leftrightarrow (\exists \nu \in t^*)(\rho \trianglelefteq \nu)$.

$(*)_4$ Let $g^{\varepsilon, \langle t_\ell \, : \, \ell \in \omega\rangle}(n)$ be the first $m > n$ as in $(*)_3$. For every $n, k \in \omega$ the sets

$$S_{n,k} = \Big\{ \bigcup_{\ell \in \omega} t_\ell \, : \, \langle t_\ell \, : \, \ell \in \omega\rangle \text{ is a branch of } T^\varepsilon_r \, \wedge g^{\varepsilon, \langle t_\ell \, : \, \ell \in \omega\rangle}(n) \leq k \Big\}$$

are open sets in the compact tree T^ε_r, and $T^\varepsilon_r = \bigcup_{k \in \omega} S_{n,k}$ is a union of an increasing sequence $\langle S_{n,k} \, : \, k \in \omega\rangle$. Hence there is K, such that $S_{n,K} = T^\varepsilon_r$ and hence $K \geq g^{\varepsilon, \langle t_\ell \, : \, \ell \in \omega\rangle}(n)$ for all branches $\langle t_\ell \, : \, \ell \in \omega\rangle$ of T^ε_r. We let $g^\varepsilon(n)$ be the minimal such K.

Now we specify the following items:

(α) We take some $g \colon \omega \to \omega$ is such that $(\forall n) g^\varepsilon(n) \leq g(n)$. Our aim is to show that η is recursive in such a g.

(β) $\varepsilon = \frac{1}{4n(*)}$, and $\varepsilon' = \frac{3}{4n(*)}$. We choose some p^* as above and some $\hat{\mathbb{Q}}$-generic filter G such that $p^* \in G$. We fix an ω branch of $T^{\varepsilon}_{p^*}$ such that $t_{g(\ell)}$ determines $\eta(\ell)$ and the part of the oracle needed for it in the sense of $(*)_3$ and $(*)_4$, and $t_{g(\ell)}(1)$ is an initial segment of a condition in G.

(γ) $p^{**} = \{\rho : \rho \in p^* \cap {}^{m(*)}2 \ \vee\ (\rho \in {}^{\omega>}2 \setminus {}^{m(*)}2 \wedge \rho \restriction m(*) \in p^*)\}$.

The proof of the following claim will finish the proof of Theorem 1.2.

Claim. *For every $n \in \omega$, $k \in \{0,1\}$, the following are equivalent:*

(i) $\eta(n) = k$,

(ii) *for some $t^1 \in T^{\varepsilon'}_{p^{**},g(n)}$ (— and this is recursive in g —) for every t^0 satisfying $t^0 \subseteq t^1$ and $t^0 \in T^{\varepsilon}_{p^{**},g(n)}$ there is $t^2 \subseteq t^0$ such that $(*)_3$ (a) + (b) holds with $t^* = \mathrm{dom}(t^2)$ and value $\eta(n) = k$.*

Proof: (i) to (ii): We assume (i). We take $t^1 = p^{**} \restriction g(n)$. If $t^0 \subseteq t^1$, $t^0 \in T^{\varepsilon'}_{p^{**},g(n)}$ is given, we may take $t^2 = t^0$. Since any branch containing t^0 and stronger than p^* forces $\eta(n) = k$, we have by the definition of $g(n)$, that the part below $g(n)$ suffices for the computation. So t^2 acts as desired.

(ii) to (i):

Assume that $\eta(n) = 1-k$. As we have "(i) $\Rightarrow$ (ii)" for this situation, there is some $s^1 \in T^{\varepsilon'}_{p^{**},g(n)}$ such that for every $s^0 \subseteq s^1$ with $s^0 \in T^{\varepsilon}_{p^{**},g(n)}$ there is $s^2 \subseteq s^0$ such that the analogues of $(*_3)$ (a) and (b) hold with $\eta(n) = 1-k$. We have t^1 as in (ii) for $\eta(n) = k$. There are q_0, q_1 witnessing $t^1, s^1 \in T^{\varepsilon'}_{p^{**},g(n)}$.

Subclaim 1: q_0, q_1 are compatible in the Amoeba forcing.

Proof of the claim: Both satisfy:

$$\begin{aligned}
&\lim(p^{**}) \supseteq \lim(q_\ell),\\
&\frac{1}{2} + \frac{1}{n(*)} \le \mathrm{Leb}(\lim(p^{**})) \le \frac{|p^* \cap {}^{m(*)}2|}{2^{m(*)}} \le \frac{1}{2} + \frac{1}{n(*)} + \frac{1}{2^{n(*)+7}},\\
&\mathrm{Leb}(\lim(q_\ell)) \ge \frac{1}{2} + \varepsilon'.
\end{aligned}$$

We show that $\mathrm{Leb}(\lim(q_0) \cap \lim(q_1)) > \frac{1}{2}$:

We have that

$$\begin{aligned}
&\mathrm{Leb}(\lim(p^{**}) \setminus (\lim(q_0) \cap \lim(q_1)))\\
&\qquad \le \mathrm{Leb}(\lim(p^{**}) \setminus (\lim(q_0))) + \mathrm{Leb}(\lim(p^{**}) \setminus \lim(q_1)))\\
&\qquad \le 2 \cdot \left(\frac{1}{4n(*)} + \frac{1}{2^{n(*)+7}}\right)\\
&\qquad = \frac{1}{2n(*)} + \frac{1}{2^{n(*)+6}},
\end{aligned}$$

hence

$$\begin{aligned}\mathrm{Leb}(\lim(q_0)\cap\lim(q_1)) \geq& \mathrm{Leb}(\lim(p^{**})) - \mathrm{Leb}(\lim(p^{**})\setminus(\lim(q_0)\cap\lim(q_1)))\\ \geq& \frac{1}{2}+\frac{1}{n(*)}-\frac{1}{2n(*)}-\frac{1}{2^{n(*)+6}} > \frac{1}{2}.\end{aligned}$$

So the sublcaim is proved.

But: q_0 and q_1 cannot be compatible in the Amoeba forcing. By the choice of p^* we have that

$$p^* \Vdash \text{“}\eta \text{ is computed by } M \text{ using the oracle } \underset{\sim}{T}.\text{”}$$

We have that $q_\ell \geq p^*$ and that $q \geq q_\ell$. But then we find $t^2_\ell \subseteq p^* \cap 2^{g(n)}$ such that

(a) $\sum_{x\in t^2_\ell} h_{p^*}(x) > \frac{1}{2}$, and

(b) if M runs on the input n and the oracle t^2_ℓ it will give the result $\eta(n)$ for $\ell = 0$ and $1-\eta(n)$ for $\ell = 1$.

Since $\eta \in V$, there cannot be two different computations, depending on two different $\underset{\sim}{T}[G] \cap {}^{g(n)}2$. Hence the assumption that q_0 and q_1 with the above properties both exist leads to a contradiction, and the Claim and Theorem 1.2 are proved.

Theorem 1.4. *Every needed real for* $\mathbf{Cof}(\mathcal{N})$ *is needed for the dominating relation.*

Proof. : Let $\{A_i : i < \kappa\}$ be a $\mathbf{Cof}(\mathcal{N})$-adequate set, such that each A_i is a Borel set. Let $\eta \in {}^\omega 2$.

For each i choose a countable elementary submodel N_i of $(\mathcal{H}(\beth_3), \in)$ to which η and A_i belong. We let G_i be a subset of $\mathbb{Q}^{N_i}$ that is generic over N_i and let $T_i = \underset{\sim}{T}[G_i]$. Now let A^*_i be

$$\begin{aligned}A^*_i = \{\rho \in {}^\omega 2 :\ &\text{no } \rho' \in {}^\omega 2 \text{ which is almost equal to } \rho\\ &\text{(i.e. } \rho(n) = \rho'(n) \text{ for every large enough } n\text{) belongs to } T_i\}\end{aligned}$$

A^*_i is a null set: We have $A^*_i = \bigcap_{n\in\omega}(\{\rho' : (\exists \rho \in T_i)\ (\rho' \restriction [n,\omega) = \rho \restriction [n,\omega))\})^c$. Furthermore we have that $\lim_{n\to\infty} \mathrm{Leb}(\{\rho' : (\exists\rho \in T)(\rho' \restriction [n,\omega) = \rho \restriction [n,\omega))\}) = 1$, because for a given ε, by the Lebesgue density Theorem (3.20 in [7]) there is some n_0 such that for $n \geq n_0$ we have for all $s \in T \cap {}^n 2$ that $\mathrm{Leb}(T \cap [s]) \cdot 2^n > 1 - \varepsilon$ and hence $\mathrm{Leb}(\{\rho' : \exists \rho \in T \rho' \restriction [n,\omega) = \rho \restriction [n,\omega)\}) > 1 - \varepsilon$.

By genericity of T_i and because $A_i \in N_i$ and because A_i is a nullset in N_i we have that $A_i \subseteq (T_i)^c$. The same argument shows that for all $s \in {}^{\omega>}2$ we have

that $\{s\hat{\ }f : \exists s' \, (|s'| = |s| \wedge s'\hat{\ }f \in A_i)\}$ is a subset of $(\lim(T_i))^c$. Hence we have that $A_i \subseteq A_i^*$. Therefore also $\{A_i^* : i < \kappa\}$ is a $\mathbf{Cof}(\mathcal{N})$-adequate set. If η is recursive in A_i^* (more precise: in one one of A_i^*'s simple codings) it is also recursive in T_i and hence by Theorem 1.2 needed for dominating. □

Fact 1.5. *We use the result of Jockusch and Solovay every real that is needed for the dominating relation is hyperarithmetic (Solovay [9]) and this is optimal (Jockusch, [6]): every hyperarithmetic real is needed for the dominating relation.*

Blass [4, Theorem 6, Corollary 8] showed that every real that is needed for $\mathbf{D}$ is also needed for $\mathbf{Cof}(\mathcal{N})$ and hence that all hyperarithmetic reals are needed for $\mathbf{Cof}(\mathcal{N})$. So this gives the other inclusion in the following corollary:

Corollary 1.6. *Exactly the hyperaritmethic reals are needed for the* $\mathbf{Cof}(\mathcal{N})$*-relation.*

2. Needed reals for the slalom relation and a general scheme

In this section we deal with a forcing $\mathbb{L}$ which is closely related to the localization forcing from [2, page 106]. Theorem 2.3 is analogous to Theorem 1.1, but for the forcing $\mathbb{L}$. Theorem 2.10 is analogous to Theorem 1.4, but the translation mechanism in the proof is different.

In the second part of the section, we collect sufficient conditions and give a general scheme for the proofs of "being computable in the generic and being in V implies being hyperarithmetic" and of "every real needed for R is Δ^1_1."

Definition 2.1.

$$\begin{aligned}\mathbb{L} = &\{p : p = (n, \bar{u}) = (n^p, \bar{u}^p), \bar{u} = \langle u_\ell : \ell \in \omega\rangle, u_\ell \in [\omega]^{\leq \ell},\\ &h(p) := \limsup\langle |u_\ell| : \ell \in \omega\rangle < \omega \text{ is well-defined}\},\end{aligned}$$

$$p \leq q \leftrightarrow \left(\bigwedge_{\ell\in\omega} u^p_\ell \subseteq u^q_\ell \wedge \bar{u}^q \restriction n^p = \bar{u}^p \restriction n^p\right).$$

The generic is considered as a characteristic function ρ with domain $\omega \times \omega$ such that $\rho(n, m) = 1 \leftrightarrow (\exists p \in G)(m \in u^p_n)$.

Notation 2.2. *An m-oracle is a function from $m \times m$ to $\{0, 1\}$. If $\bar{u} = \langle u_\ell : \ell < m\rangle$, $u_\ell \in [\omega]^{<\ell}$ the $\bar{u}$-oracle $\rho_{\bar{u}} \in {}^{m\times m}2$ is defined by $\rho_{\bar{u}}(n_1, n_2) = 1 \leftrightarrow n_2 \in u_{n_1}$. We allow that $(\exists \ell < m)\ \max(u_\ell) > \lg(\bar{u}) = m$.*

Theorem 2.3. *Assume that M is a Turing machine and that $\eta \in {}^{\omega}2$. Let $\underset{\sim}{G}$ be a name for an $\mathbb{L}$-generic element. Suppose that $p^* \in \mathbb{L}$ and that*

$$p^* \Vdash_{\mathbb{L}} M \text{ computes } \eta \text{ from } \underset{\sim}{G}.$$

Then η is hyperarithmetic.

Proof. Let $n^* = n^{p^*}$ and $\bar{u}^* = \bar{u}^{p^*} \upharpoonright n^*$, and $h^* = h(p^*)$. By a density argument we may assume that $n^* > 4h^* \wedge (\forall \ell)(\ell \geq n^* \rightarrow |u^*_\ell| \leq h^*)$.

We let

$$\begin{aligned} T = T_{\bar{u}^*} = \{\bar{u} \,:\, & n^* \leq m < \omega, \bar{u} = \langle u_\ell \,:\, \ell \in m\rangle, u_\ell \in [\omega]^{\leq \ell}, \\ & \bar{u} \upharpoonright n^* = \bar{u}^* \wedge \ell \geq n^* \rightarrow |u_\ell| = h^*\}. \end{aligned}$$

We order T by the initial segment relation $\trianglelefteq$. The set of all infinite branches of T is $[T] = \{\bar{u} \,:\, \forall n \bar{u} \upharpoonright n \in T\}$.

If $\bar{u} \in T_{\bar{u}^*}$ we let

$$\begin{aligned} \Xi_{\bar{u}} = \Xi_{\bar{u},\bar{u}^*} = \{\rho_{\bar{v}} \,:\, & \lg(\bar{u}) = \lg(\bar{v}), \bar{v} \upharpoonright n^* = \bar{u}^*, \\ & (n^* \leq \ell < \lg(\bar{u}) \rightarrow [0, \lg(\bar{u})) \cap u_\ell \subseteq v_\ell)\}. \end{aligned}$$

Fact 2.4. *For every $j < \omega$, $\bar{u} = \langle u_\ell \,:\, \ell \in \omega\rangle \in [T_{\bar{u}^*}]$, such that for each ℓ, and $\lg(u_\ell) = \ell$, there are $m \in [n^*, \omega)$, $\bar{v} \in \Xi_{\bar{u}\upharpoonright m} \cap \Xi_{\bar{u}^*}$ such that*

$$(*) \qquad \begin{aligned} & \textit{with } \rho_{\bar{v}} \textit{ as an oracle on domain } \lg(\bar{v}) \times \lg(\bar{v}), M \textit{ finishes its run} \\ & \textit{and gives the result } \eta(j) \end{aligned}$$

Proof. The conditions $(n^*, \bar{u})$ and $p^* = (n^{p^*}, \bar{u}^{p^*})$ are compatible: $(n^*, \bar{v}) = (n^*, \langle u_\ell \cup u^{p^*}_\ell \,:\, \ell \in \omega\rangle) \in \mathbb{L}$ is stronger or equal to both of them (here we use $n^* > 4h^*$) and in $\Xi_{\bar{u}\upharpoonright m}$ for all m. We take a generic to which $(n^*, \bar{v})$ belongs. Consider the run of M, it uses only $\bar{v} \cap (m \times m)$ for m large enough. $\square$

Fact 2.5. *For every $j < \omega$ there is $m_j \in (n^*, \omega)$ such that such that for every $\langle u_\ell \,:\, \ell \in \omega\rangle \in [T_{\bar{u}^*}]$, there is $\rho_{\bar{v}} \in \Xi_{\bar{u}\upharpoonright m_j} \cap \Xi_{\bar{u}^*}$ such that $(*)$ holds.*

Proof. By the previous lemma and by König's lemma. All the levels of $T_{\bar{u}^*}$ are finite. Note that $\Xi_{\bar{u}}$ depends only on $\langle u_\ell \cap \lg(\bar{u}) \,:\, \ell < \lg(\bar{u})\rangle$. $\square$

Definition 2.6. *$g_{M,\bar{u}^*} \in {}^{\omega}\omega$ is defined by*

$$g_{M,\bar{u}^*}(j) = \min\{m_j \,:\, m_j \textit{ in as in the Fact 2.5}\}.$$

Claim 2.7. *For every $j \in \omega, k < 2$ and $m \geq g_{M,\bar{u}^*}(j)$ the following are equivalent:*

(i) *$\eta(j) = k$,*

(ii) *for some $\bar{u} = \langle u_\ell : \ell < m\rangle$ and h^*, such that $(\ell \in [n^*, m) \to u_\ell \in [m]^{\leq h^*})$, $\bar{u} \restriction n^* = \bar{u}^*$ for every $\bar{u}' = \langle u'_\ell : \ell < m\rangle$ such that $\ell \in [n^*, m) \to u'_\ell \in [m]^{\leq h^*}$, $\bar{u}' \restriction n^* = \bar{u}^*$ there is $\bar{v} \in \Xi_{\bar{u}} \cap \Xi_{\bar{u}'} \subseteq {}^{m\times m}2$ such that M running with oracle $\rho_{\bar{v}}$ and input j finishes its run and gives the result k.*

Proof. : (i) $\Rightarrow$ (ii): By the previous fact, $\bar{u}^{p^*} \restriction m$ is as required. (ii) $\Rightarrow$ (i): Let $\bar{u}$ be as guaranteed in (ii). It is said there "for every $\bar{u}'$" so in particular for $\bar{u}' = \bar{u}^{p^*} \restriction m$, there is $\rho \in \Xi_{\bar{u}} \cap \Xi_{\bar{u}'}$ as there. Now we can find a condition $q \in \mathbb{L}$ such that $n^q = m > n^*$, $\bar{u}^q \restriction n^* = \bar{u}^*$, $n^* \leq \ell < m \Rightarrow u^q_\ell = u^{p^*}_\ell \cup v_\ell = u'_\ell \cup v_\ell$, $\ell \geq m \to u^q_\ell = u^{p^*}_\ell$. So

(α) $p^* \leq q$ and $q \Vdash \underset{\sim}{G} \restriction {}^{m\times m}2 = \rho_{\bar{v}}$, hence

(β) $q \Vdash$ "M running with the oracle $\underset{\sim}{G}$ and input j gives the result k", and recall

(γ) $p^* \Vdash$ "M computes η".

By $(\alpha) + (\beta) + (\gamma)$ we get that $\eta(j) = k$ is as required. $\square$

Conclusion 2.8. *Assume that $\eta \in {}^\omega\omega$, $\eta \in N$, G is $\mathbb{L}$-generic over N and that $\underset{\sim}{\rho}[G] = \rho$ and $N[G] \models$ "$\eta \leq_T \rho$". Then η is hyperarithmetic.*

Proof. Analogous to the proof of 1.2 for N instead of V. We use 2.7.

Definition 2.9. *$S \in {}^\omega({}^{\omega>}[\omega])$ is called a slalom iff for all n, $|S(n)| \leq n$.*

Theorem 2.10. *Exactly the hyperarithmetic reals are needed for the slalom relation*

$$\mathbf{SL} = \{(f, S) : f \in {}^\omega\omega \wedge S \text{ is a slalom and } (\forall n \in \omega)(f(n) \in S(n))\}.$$

Proof. First show that only hyperarithmetic reals are needed for $\mathbf{SL}$: Let $\{S_i : i < ||SK||\}$ be an SL-adequate set. Let $\eta \in {}^\omega 2$. We take $N_i \prec (H(\beth_3), \in)$ such that $\eta, S_i \in N_i$. Then we let G_i be $\mathbb{L}$-generic over N_i. Now we set $S^*_i = \{\rho : (\exists \rho' \in G_i)\rho' =^* \rho\}$. Then we have that $S_i \subseteq S^*_i$, S^*_i is the union of ω slaloms, each of them computable from G_i, and the members of all the unions form an $\mathbf{SL}$-adequate set.

All hyperarithmetic reals are needed for $\mathbf{SL}$, because all of then are needed for $\mathbf{D}$. Suppose that $\{\langle S^\alpha_i : i \in \omega\rangle : \alpha \in ||\mathbf{SL}||\}$ is $\mathbf{SL}$-adequate and that $\eta \in {}^\omega 2$ is hyperarithmetic. Then $\{\langle \max S_i : i \in \omega\rangle : \alpha \in ||\mathbf{SL}||\}$ is $\mathbf{D}$-adequate and hence there is some element f in it from which η is computable. But then

of course η is also computable in any slalom where f stems from. $\square$

From our two examples $(\mathbb{Q}, \mathbf{Cof}(\mathcal{N}))$ and $(\mathbb{L}, \mathbf{SL})$ we collect the following scheme:

Theorem 2.11. *Assume that*

(a) $T \subseteq H(\aleph_0)$ *is recursive,* T *is a tree with* ω *levels and each level is finite, each* $v \in T$ *is a finite function from* $H(\aleph_0)$ *to* $H(\aleph_0)$.

(b) Q *is a forcing notion, and* ρ_n, $n \in \omega$, *are* Q*-names, and*

$$\Vdash_Q (\forall n \in \omega)\ (\rho_n \in \lim(T))\ \wedge\ (\forall x \in \mathrm{range}(R)) \bigvee_{n \in \omega} \forall y (yRx \rightarrow yR\rho_n).$$

(c) *For each* $n \in \omega$ *we have: For a dense set of* $p_0 \in Q$ *there is some* $p \geq p_0$ *such that the following conditions are fulfilled:*

- *(α)* *Let* $T_{n,p} = \{\nu \in T : p \Vdash \nu \subseteq \rho_n\}$. *This is a subtree of* T.
- *(β)* *Let* $S^*_{n,p} = \Big\{t$: *for some subtree* T' *of* $T_{n,p}$ *and some* k, $t = \{\nu \in T' : \mathit{level}_{T_{n,p}}(\nu) \leq k\}$, *and no maximal node of* t *has level* $k\Big\}$, *and order* $S^*_{n,p}$ *naturally.*
- *(γ)* $S_{n,p}$ *is a recursive subtree of* $S^*_{n,p}$ *such that*
  - *(i)* $T_{n,p}$ *is an* ω*-branch of* $S_{n,p}$,
  - *(ii)* *for every branch* $\bar{t} = \langle t_\ell : \ell \in \omega \rangle$ *of* $S_{n,p}$ *there is* $q \in Q$ *such that* q *is compatible with* p *and* $T_{n,q} = \bigcup_{\ell \in \omega} t_\ell$.

(d) $\eta \in {}^\omega 2$ *or* ${}^\omega \omega$

Then we have for every $n \in \omega$*: if* $\Vdash_Q$ *"*η *is recursive in* $\underset{\sim}{\rho_n}$*" then* η *is hyperarithmetic.*

Proof. So for some p^* as in (c) and Turing machine M

$$p^* \Vdash_Q \text{“}M \text{ computes } \eta \text{ from } \underset{\sim}{\rho_n}\text{”}.$$

Let S_{n,p^*} and S^*_{n,p^*} be as in clause (c). Now we prove some intermediate facts, and the proof of 2.11 will be finished with 2.15.

Fact 2.12. *For every* ω*-branch* $\langle t_k : k \in \omega \rangle$ *of* S_{n,p^*} *and* $j \in \omega$ *for some (= every) large enough* $k \in \omega$ *for some* $\nu \in t_k \cap \mathit{level}_k(T_{n,p^*})$ *if* M *runs on input* j *and oracle* ν *it finishes (so we do not ask oracle questions outside the domain) and gives the result* $\eta(j) = k$.

Proof. There is q compatible with p^* such that $T_{n,q} \subseteq \bigcup_{n \in \omega} t_n$. Let $r \geq p^*, q$, and let $G \subseteq Q$ be generic with $r \in G$, so $p^* \in G$. If M runs with $\underset{\sim}{\rho_n}[G]$ it gives

$\eta(j)$, so for some $\nu \in T$, $\nu \subseteq \underset{\sim}{\rho}_n[G]$. And M can use as an oracle only ν, but as $q \in G$, $\nu \in T_{n,q} \subseteq \bigcup_{\ell\in\omega} t_\ell$. Of course any ν', $\nu \subseteq \nu' \in T_{n,p^*}$ can serve.

Fact 2.13. *For $j \in \omega$, for every large enough m, for every $t \in level_m(S_n)$ there is $\nu \in t \cap level_m(T_{n,p^*})$ such that if M runs with ν as an oracle then it computes $\eta(j)$.*

Proof. By the previous fact and König's lemma.

Definition 2.14. *We define $g_{p^*} \in {}^\omega\omega$ by $g_{p^*}(j) = \min\{m : m$ as in $2.13\}$.*

Crucial Fact 2.15. *For $j, n \in \omega$, $k \in 2$, the following are equivalent for any $m \geq g_{p^*}(j)$:*

- *(i)* $\eta(j) = k$.
- *(ii)* *there is $t^1 \in level_m(S_{n,p^*})$ such that for every $t^2 \in level_m(S_{n,p^*})$ there is $\nu \in t^1 \cap t^2$ such that if we let run M with input j and oracle ν then the run finishes and there are no questions to the oracle that do not have an answer, and it gives answer k.*

Proof. Analogous to 2.7 $\square_{2.11}$

Remark 2.16. *1. Usually, S_{n,p^*} is not so dependent on p^*, rather we have that $Q = \bigcup_{k\in\omega} Q_k$, and for all $k \in \omega$ we have S_{n,p^*} as above being the same for each $p^* \in Q_k$.*

2. Actually we use in (c)(γ)(i) only $T_{n,q} = \bigcup_{k\in\omega} t_k$. But we use $T_{n,p^*} = \bigcup_{k\in\omega} t_k$ for some ω-branch.

Theorem 2.17. *A sufficient condition for "every real needed for R is Δ^1_1" is: For some forcing notion Q and some Q-names $\underset{\sim}{\rho}_n$, $n \in \omega$, we have*

- *(a)* $\Vdash_Q$ "$\underset{\sim}{\rho}_n \in [T], \underset{\sim}{\rho}_n \in \mathrm{range}(R)$"
- *(b)* $\Vdash_q$ *"for every $x \in \mathrm{dom}(R)$ for some n, $xR\underset{\sim}{\rho}_n$"*
- *(c)* *for each n: Q, T and $\underset{\sim}{\rho}_n$ satisfy the conditions in 2.10 or just its conclusion.*

Proof. Like the first half of the proof of Theorem 2.10.

3. Weakly needed reals for the reaping relation

In this section we show that it is consistent that all hyperarithmetic reals are weakly needed for the reaping relation. In Section 5 we shall prove in ZFC that not all hypearithmetic real are are needed for the reaping relation,

answering another question from Blass' work [4]. In a model of CH, the notions "needed real" and "weakly needed real" coincide, and thus in such a model not all hyperarithmetic reals are weakly needed for the reaping relation. The model of this section, together with the result from Section 5, gives an example for the fact that in contrast to the notion of "being needed", the notion of "being weakly needed" is not absolute.

Definition 3.1. *The relation*

$$\mathbf{R} = \{(f, X) \,:\, f \in {}^{\omega}2, X \in {}^{\omega}[\omega] \wedge f \restriction X \text{ is constant}\}$$

is called the reaping or the refining or the unsplitting relation. We say "X refines f" if $f \restriction X$ is constant. We say "$\mathcal{R}$ refines f" if there is some $X \in \mathcal{R}$ that refines f. Finally we say "$\mathcal{R}$ refines F" if for every $f \in F$ we have that $\mathcal{R}$ refines f.

The norm of this relation is called $\mathfrak{r}$, the reaping number or the refining number or the unsplitting number.

Definition 3.2. *Let $g \in {}^{\omega}\omega$ be strictly increasing and $g(n) > n$.*

(1) *We say $A \in [\omega]^{\omega}$ is g-slow if $(\exists^{\infty} n)|A \cap g(n)| \geq n$.*

(2)

$$\mathcal{F}_g = \{f \,:\, \mathrm{dom}(f) \in [\omega]^{\omega},\ \text{for } i \in \mathrm{dom}(f) \text{ we have that } f(i) = (f^1(i), f^2(i)) \\ \text{and } f^2(i) \in [g(f^1(i))]^{\geq f^1(i)} \text{ and } \limsup\langle f^1(i) \,:\, i \in \mathrm{dom}(f)\rangle = \omega\}.$$

(3) *We say that $\bar{A}$ is (g, κ)-o.k. if*

  (a) $\bar{A} = \langle A_i \,:\, i < \kappa\rangle$, *and*

  (b) $A_i \in [\omega]^{\omega}$,

  (c) *if $k < \omega$, $f_0, \dots, f_{k-1} \in \mathcal{F}_g$, $\bigcap_{\ell \in \omega} \mathrm{dom}(f_\ell) = B \in [\omega]^{\omega}$ and $\limsup\langle \min\{f^1_\ell(i) \,:\, \ell \in k\} \,:\, i \in B\rangle = \omega$, then for some $\alpha = \alpha(\langle f_\ell \,:\, \ell < k\rangle)$ we have that:*

  For every $u_\ell \in [\kappa \setminus \alpha]^{<\omega}$ and $\eta_\ell \in {}^{u_\ell}2$ the set

$$\{n \in B \,:\, (\forall \ell < k)(f^2_\ell(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset)\} \tag{3.1}$$

  is infinite,

  where

$$\bar{A}^{[\eta_\ell]} = \bigcap_{i \in u_\ell} A_i^{\eta_\ell(i)}, \text{ and}$$

$$A_i^{\ell} = \begin{cases} A_i, & \text{if } \ell = 1, \\ \omega \setminus A_i, & \text{if } \ell = 0. \end{cases}$$

Remark: $f \in \mathcal{F}_g$ implies that $\bigcup_{i \in \mathrm{dom}(f)} f^2(i)$ is not g-slow.

Claim 3.3. *We get an equivalent notion to "$\bar{A}$ is (g,κ)-o.k.", if we modify the Definition 3.2(c) as in (a) and/or as in (b), where*

(a) *We demand 3.2(c) only for $f_\ell \in \mathcal{F}_g$ that additionally satisfy $\mathrm{dom}(f_0) = \dots = \mathrm{dom}(f_{k-1}) = \omega$.*

(b) *We demand 3.2(c) only for $f_0, \dots, f_{k-1} \in \mathcal{F}_g$ such that $\langle \min\{f^1_\ell(i) \,:\, i < k\} \,:\, i < B\rangle$ is strictly increasing (we can even demand, increasing faster than any given h), and for $i \in B$, $\max\{f^1_\ell(i) \,:\, \ell < k\} < \min\{f^1_\ell(i+1) \,:\, \ell < k\}$.*

Proof. (a) Suppose the $f_0, \dots, f_{k-1} \in \mathcal{F}_g$ in the original sense, and that we have required the analogue of 3.2(c) only for $\mathcal{F}_g$ in the restricted sense. We suppose that $\bigcap_{\ell<k} \mathrm{dom}(f_\ell) = B$ and take a strictly increasing enumeration $\{b_r \,:\, r \in \omega\}$ of B. Then we take $\tilde{f}_\ell \colon \omega \to [\omega]^{<\omega}$, $\tilde{f}_\ell(r) = f_\ell(b_r)$ for $r \in \omega$. The analogue of 3.2 for the $\mathcal{F}_g$ in the restricted sense gives $\alpha \in \kappa$ and infinite intersections in (3.1) for the $\tilde{f}_\ell$. The intersections are also infinite for the original f_ℓ.

(b) Suppose that $k < \omega$, $f_0, \dots, f_{k-1} \in \mathcal{F}_g$, $\bigcap_{\ell\in\omega} \mathrm{dom}(f_\ell) = B \in [\omega]^\omega$ and $\limsup \langle \min\{m_{f_\ell(i)} \,:\, \ell \in k\} \,:\, i \in B\rangle = \omega$. Then we can thin out the domain B to some infinite B', inductively on i such that the $f_\ell \restriction B'$ fulfil all the requirements from 3.3(b).

Crucial Fact 3.4. *Let $g \in {}^\omega\omega$. If $\mathfrak{r} < \kappa = \mathrm{cf}(\kappa)$ and if there is some $\bar{A}$ that is (g,κ)-o.k., then every Δ^1_1-real that is computable in every function $g' \geq^* g$ is weakly needed for the refining relation.*

Proof. Let $\mathcal{R} = \{B_\alpha \,:\, \alpha < |\mathcal{R}|\}$ witness $\mathfrak{r} < \kappa$. The family $\bar{A}$ is refined by $\mathcal{R}$: For $i < \kappa$ for some $\alpha_i < |\mathcal{R}|$ and $\nu(i) \in \{0,1\}$ we have that $B_{\alpha_i} \subseteq A_i^{\nu(i)}$. Since κ is regular and since $\mathfrak{r} < \kappa$, there are for some $\ell < 2$ and some $\beta < |\mathcal{R}|$ such that

$$Y = \{i < \kappa \,:\, \nu(i) = \ell \wedge \alpha_i = \beta\}$$

is unbounded. So $B_\beta \subseteq \bigcap_{i\in Y} A_i^{\nu(i)}$. We claim that B_β is not g-slow. Why? Otherwise we have $C = \{n < \omega \,:\, |B_\beta \cap g(n)| > n\} \in [\omega]^\omega$, and we may take $f \in \mathcal{F}_g$ such that $C = \mathrm{dom}(f)$, $f^1(n) = n$ and $f^2(n) = B_\beta \cap g(n)$. Take any $\alpha \in \kappa$. Then we take u_0 such that $u_0 = \{\gamma\}$, $\gamma \in Y$, $\gamma > \alpha$ and $\eta_0 = \{(\gamma, 0)\}$ and $\eta'_0 = \{(\gamma, 1)\}$. Then we do not have $(\exists^\infty n) f^2(n) \cap A^0_\gamma \neq \emptyset$ and $(\exists^\infty n) f^2(n) \cap A^1_\gamma \neq \emptyset$ at the same time, because B_β is refining A_γ. So $\bar{A}$ is not (g,κ)-o.k., in contrast to our assumption.

But now we can compute recursively from B_β some $g' \geq^* g$, for example we may take $g'(n) =$(the nth element of B_β) $+1$. Hence every hyperarithmetic real

that is computable in every function $g' \geq^* g$ is recursive in B_β. $\square$

So, how do we get the premises of the crucial fact? The rest of this section will be devoted to this issue. We take g growing sufficiently fast so that every Δ^1_1-function is computable in every $g' \geq g$. Such a g exists by [6, 9] and the fact that there are only countably many Δ^1_1-functions. We fix such a g. We consider the case $\kappa = \mathrm{cf}(\kappa) > \aleph_1$ and intend to show the consistency of "$\mathfrak{r} = \aleph_1$ and there is some $\bar{A}$ that is (g, κ)-o.k."

Definition 3.5. *(1)* $K_g = K = \{(P, \underset{\sim}{\bar{A}}) : P$ *is a ccc forcing and* $\Vdash_P$ "$\underset{\sim}{\bar{A}}$ *is* (g, κ)*-o.k."*$\}$. *For a fixed* g*, we often leave out the subscript.*

(2) $(P_1, \underset{\sim}{\bar{A}_1}) \leq_K (P_2, \underset{\sim}{\bar{A}_2})$ *iff* $P_1 \lessdot P_2$ *and* $\underset{\sim}{\bar{A}_1} = \underset{\sim}{\bar{A}_2}$.

Claim 3.6. *(1) We have that* $K \neq \emptyset$*. In fact, if* P *is the forcing adding* κ *Cohen reals and* $\underset{\sim}{\bar{A}}$ *is the enumeration of the* κ *Cohen reals, then* $(P, \underset{\sim}{\bar{A}}) \in Kg$ *for any function* g*. (This is true for any function* g*.)*

(2) If $(P_\alpha, \underset{\sim}{\bar{A}}) \in K$ *for* $\alpha < \delta$*,* δ *a limit cardinal, and* $\langle P_\alpha : \alpha < \delta\rangle$ *is increasing and continuous, and* $P = \bigcup_{\alpha<\delta} P_\alpha$*, then* $(P, \underset{\sim}{\bar{A}}) \in K$ *and* $\alpha < \delta \Rightarrow (P_\alpha, \underset{\sim}{\bar{A}}) \leq_K (P, \underset{\sim}{\bar{A}})$.

Proof. (1) Suppose that $f_0, \dots, f_{k-1} \in V[G_\kappa]$ are injective functions. We take α such that $f_0, \dots, f_{k-1} \in V[G_\alpha]$ where G_α is a generic filter for the first α Cohen reals. Suppose that $\eta_\ell \in {}^{u_\ell}2$. Now a density argument gives that these $\bar{A}^{[\eta_\ell]}$ "flip for infinitely many $n \in B$" to 0 or to 1 within $f^2_\ell(n)$ for every $\ell < k$.

(2) P has the c.c.c. by a Fodor argument. Now we show that $\Vdash_P$ "$\underset{\sim}{\bar{A}}$ is (g, κ)-o.k."$\}$. Only the case of $\mathrm{cf}(\delta) = \omega$ is not so easy. We suppose that $\delta = \bigcup_{n\in\omega} \alpha(n)$, $0 < \alpha(n) < \alpha(n+1)$. Towards a contradiction we assume that $p^* \in P_{\alpha(0)}$, and

$$p^* \Vdash \text{"}\underset{\sim}{B}, \langle \underset{\sim}{f_\ell} : \ell < k\rangle \text{ form a counterexample to } \bar{A} \text{ being } (g, \kappa)\text{-o.k."}$$

For each $n \in \omega$ we find $\langle q_{n,i} : i \in \omega\rangle$ such that

(α) $q_{n,i} \in P$,

(β) $q_{n,0} = p^*$,

(γ) $P \models q_{n,i} \leq q_{n,i+1}$,

(δ) for some $\underset{\sim}{b_{n,i}}$, $\underset{\sim}{f^1_{n,\ell,i}}$, $\underset{\sim}{f^2_{n,\ell,i}}$ $P_{\alpha(n)}$-names we have

$$q_{n,i} \Vdash \text{"}\underset{\sim}{b_{n,i}} \text{ is the } i\text{-th member of } \underset{\sim}{B}, \underset{\sim}{f_\ell}(\underset{\sim}{b_{n,i}}) = (\underset{\sim}{f^1_{n,\ell,i}}, \underset{\sim}{f^2_{n,\ell,i}})\text{"},$$

(ε) $q_{n,i} \restriction \alpha(n) = q_{n,0} \restriction \alpha(n) = p^* \restriction \alpha(n)$.

How do we choose these? Let n and $\alpha(n)$ be given. Then we choose $q'_{n,i}$ increasing in i such that $q'_{n,i} \in P$ and $b'_{n,i}$, $(f^1)'_{n,i}$, $(f^2)'_{n,\ell,i}$ in V and

$$q'_{n,i} \Vdash \bigwedge_{\ell<k} \text{the } i\text{th element of } \underset{\sim}{B} = \check{b'_{n,i}} \wedge \underset{\sim}{f}_\ell(\check{b'_{n,i}}) = ((\check{f^1)'_{n,\ell,i}}, (\check{f^2)'_{n,\ell,i}}).$$

Then we take

$$\begin{aligned} \underset{\sim}{b}_{n,i} &= (b'_{n,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ \underset{\sim}{f}^1_{n,\ell,i} &= ((f^1)'_{n,\ell,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ \underset{\sim}{f}^2_{n,\ell,i} &= ((f^2)'_{n,\ell,i}, q'_{n,i} \restriction P_{\alpha(n)}), \\ p_{n,i} &= p^* \restriction \alpha(n) \cup q'_{n,i} \restriction [\alpha(n), \delta). \end{aligned}$$

Here, the restriction $\restriction \alpha$ is any reduction function witnessing $P_\alpha \lessdot P$ (see [1]), and in the general case, if P_α is not the initial segment of length α of some iteration, the term $q'_{n,i} \restriction [\alpha(n), \delta)$ has to be interpreted as some element from a quotient forcing algebra.

Now for every n we define $P_{\alpha(n)}$-names

$$\begin{aligned} \underset{\sim}{B}'_n &= \{\underset{\sim}{b}_{n,i} : i < \omega\}, \\ \underset{\sim}{f}_{\ell,n} &\colon \underset{\sim}{B}'_n \to V, \\ \underset{\sim}{f}_{\ell,n}(\underset{\sim}{b}_{n,i}) &= (\underset{\sim}{f}^1_{\ell,n}(\underset{\sim}{b}_{n,i}), \underset{\sim}{f}^2_{\ell,n}(\underset{\sim}{b}_{n,i})) = (\underset{\sim}{f}^1_{\ell,n,i}, \underset{\sim}{f}^2_{\ell,n,i}). \end{aligned}$$

Now we have that

$$\begin{aligned} p^* \Vdash \text{“}&\underset{\sim}{B}'_n \in [\omega]^{\aleph_0}, \underset{\sim}{f}_{\ell,n} \text{ is a function with domain } \underset{\sim}{B}'_n \text{ and} \\ &\limsup \langle \underset{\sim}{f}^1_{\ell,n}(b) : b \in \underset{\sim}{B}'_n \rangle = \omega \text{ and} \\ &\underset{\sim}{f}^2_{\ell,n,i} \text{ when defined is a subset of } [0, g(\underset{\sim}{f}^1_{\ell,n,i})) \text{ of cardinality } > \underset{\sim}{f}^1_{\ell,n,i}\text{”}. \end{aligned}$$

As $(P_{\alpha(n)}, \underset{\sim}{\bar{A}})$ is in K we for every n

$$\begin{aligned} p^* \restriction \alpha(n) \Vdash_{P_{\alpha(n)}} \text{“ for some } \underset{\sim}{\beta} < \kappa \text{ for every } u_\ell \subseteq [\kappa \setminus \underset{\sim}{\beta}]^{\aleph_0} \text{ for every } \eta_\ell \in {}^{u_\ell}2 \\ \left\{ b \in \underset{\sim}{B}'_n : \bigwedge_{\ell<k} \underset{\sim}{f}^2_{\ell,n}(b) \cap \underset{\sim}{\bar{A}}^{[\eta_\ell]} \neq \emptyset \right\} \text{ is infinite.”} \end{aligned}$$

Let $\underset{\sim}{\beta}_n < \kappa$ be such a $P_{\alpha(n)}$-name. Since $P_{\alpha(n)}$ has the ccc, there is some $\beta^*_n < \kappa$ such that $\Vdash_{P_{\alpha(n)}} \underset{\sim}{\beta}_n < \beta^*_n < \kappa$. Since κ is regular we have that $\beta^* = \bigcup_{n \in \omega} \beta^*_n < \kappa$.

It suffices to prove that

$$p^* \Vdash \text{“}\beta^* \text{ is as required in the definition of } (g, \kappa)\text{-o.k.”}$$

If not, then there are counterexamples $u_\ell \in [\kappa \setminus \beta^*]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$, q and b^* such that

$$(\diamond) \qquad \begin{aligned} &p^* \leq q \in P = P_\delta \\ &q \Vdash \text{“}\Big\{b \in \underset{\sim}{B} \,:\, (\forall \ell < k)(\underset{\sim}{f}^2_\ell(b) \cap \bar{\underset{\sim}{A}}^{[\eta_\ell]} \neq \emptyset)\Big\} \subseteq [0, b^*]\text{”}. \end{aligned}$$

For some $n(*) < \omega$ we have that $q \in P_{\alpha(n(*))}$. Let $G \subseteq P$ be generic over V, and let $q \in G_{\alpha(n(*))}$. So by the choice of $\beta_{n(*)} < \beta^*$ we have that

$$p \Vdash_{P_{\alpha(n(*))}} C = \{b \in \underset{\sim}{B}'_{n(*)} \,:\, (\forall \ell < k)(\underset{\sim}{f}^2_{\ell,n(*)}(b) \cap \bar{\underset{\sim}{A}}^{[\eta_\ell]} \neq \emptyset)\} \text{ is infinite”}.$$

Recall that $\underset{\sim}{B}'_{n(*)}$ and $\underset{\sim}{f}_{\ell,n(*)}(b)$ are $P_{\alpha(n(*))}$-names and that $\bar{\underset{\sim}{A}}^{[\eta_\ell]}$ is a P_0-name.

Now $\underset{\sim}{B}'_{n(*)} = \{\underset{\sim}{b}_{n(*),i} \,:\, i < \omega\}$, so for some i we have that $\underset{\sim}{b}_{n(*),i}[G] > b^*$. So $q_{n(*),i} \in G \cap P_{\alpha(n(*))}$ forces “the i-th member of $\underset{\sim}{B}$ is $\underset{\sim}{b}_{n(*),i}$ and $\underset{\sim}{f}_\ell(\underset{\sim}{b}_{n(*),i}) = \underset{\sim}{f}_{\ell,n(*)}(\underset{\sim}{b}_{n(*),i}) = (\underset{\sim}{f}^1_{\ell,n(*),i}, \underset{\sim}{f}^2_{\ell,n(*),i})$. Note that $q_{n(*),i} \restriction \alpha(n(*)) = p^* \restriction \alpha(n(*))$ according to ε), and hence $q_{n(*),i} \not\perp q$. So there is some $r \geq q$ and $r \geq q_{n(*),i}$. Such an r forces the contrary of the property forced in $(\diamond)$, and finally we reached a contradiction. $\square$

Now 3.7 and 3.8 are like [8]. For $h \colon \omega \to \omega$ We write $\lim_D \langle h(i) \,:\, i \in \omega \rangle = \omega$ if for all $m < \omega$ we have that $\{i \,:\, h(i) > m\} \in D$.

Claim 3.7. *Assume that in V:*

(a) $\bar{A}$ *is (g, κ)-o.k.*

(b) $\kappa = 2^{\aleph_0}$.

Then there is an ultrafilter D on ω such that

$$(*) \qquad \begin{aligned} &\textit{if } f \in \mathcal{F}_g \textit{ and } \mathrm{dom}(f) \in D \textit{ and } \lim_D \langle f^1(i) \,:\, i \in \mathrm{dom}(f) \rangle = \omega \\ &\textit{then for some } \alpha_f < \kappa \textit{ for every } u \in [\kappa \setminus \alpha_f]^{<\aleph_0} \textit{ and } \eta \in {}^u2 \\ &\textit{we have that } \{n \in \mathrm{dom}(f) \,:\, f^2(n) \cap \bar{A}^{[\eta]} \neq \emptyset\} \in D. \end{aligned}$$

Proof. Let $\mathcal{F}_g = \{f_j \,:\, j < \kappa\}$. Let $\mathcal{AP}$ be the set of tuples (D, i, α) such that

(i) D is a filter on ω containing the co-finite subsets, $\emptyset \notin D$, $i, \alpha < \kappa$,

(ii) D is generated by $< \kappa$ members,

(iii) if $k < \omega$ and for $\ell < k$, $j_\ell < i$, and $\mathrm{dom}(f_{j_\ell}) \in D$ and $\lim_D \langle f^1_{j_\ell}(i) \,:\, i \in \mathrm{dom}(f_{j_\ell}) \rangle = \omega$ and $u_\ell \in [\kappa \setminus \alpha]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$, then

$$\Big\{n \in \bigcap_{\ell < k} \mathrm{dom}(f_{j_\ell}) \,:\, \bigwedge_{\ell < k} \Big(f^2_{j_\ell}(n) \cap \bar{A}^{[\eta]} \neq \emptyset\Big)\Big\} \neq \emptyset \bmod D.$$

Let $(D_1, i_1, \alpha_1) \leq_{\mathcal{AP}} (D_2, i_2, \alpha_2)$ if both tuples are in $\mathcal{AP}$ and

(α) $D_1 \subseteq D_2$, $i_1 \le i_2$, $\alpha_1 \le \alpha_2$, and

(β) if $k < \omega$ and $\{j_0, \dots, j_{k-1}\} \subseteq i_1$, $\operatorname{dom}(f_{j_\ell}) \in D_2$ and $\lim_{D_2} \langle f^1_{j_\ell}(i) : i \in \operatorname{dom}(f_{j_\ell})\rangle = \omega$ and $u_\ell \subseteq [\alpha_1, \alpha_2)$ is finite and $\eta_\ell \in {}^{u_\ell}2$ then

$$\left\{ n \in \bigcap_{\ell<k} \operatorname{dom}(f_{j_\ell}) : \bigwedge_{\ell<k} f^2_{j_\ell(n)} \cap \bar{A}^{[\eta_\ell]} \ne \emptyset \right\} \in D_2.$$

Now we have that

$\boxtimes_1$ $(\mathcal{AP}, \le_{\mathcal{AP}})$ is a non-empty partial order. Take $i = \alpha = 0$ and D the filter of all cofinite subsets of ω.

$\boxtimes_2$ In $(\mathcal{AP}, \le_{\mathcal{AP}})$ every increasing sequence of length $< \kappa$ has an upper bound, namely, take the filter generated by the union in the first coordinate and take the supremum in the second and in the third coordinate.

$\boxtimes_3$ If $B \subseteq \omega$ and $(D, i, \alpha) \in \mathcal{AP}$ then there are some D', i', α' such that $(D', i', \alpha') \ge_{\mathcal{AP}} (D, i, \alpha)$ and that $B \in D'$ or that $\omega \setminus B \in D'$. Why? Try $D' =$ the filter generated by $D \cup \{B\}$ and the same i and α. If this fails then we can find $k < \omega$, such that for $\ell < k$ we have $j_\ell < i$, such that $\operatorname{dom}(f_{j_\ell}) \in D'$ and $\lim_{D'} \langle f^1_{j_\ell}(i) : i \in \operatorname{dom}(f_{j_\ell})\rangle = \omega$, $u_\ell \in [\kappa \setminus \alpha]^{<\aleph_0}$, $\eta_\ell \in {}^{u_\ell}2$ and such that

$$\left\{ n \in \bigcap_{\ell<k} \operatorname{dom}(f_{j_\ell}) : f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \ne \emptyset \right\} \cap B = \emptyset \bmod D.$$

Let $\alpha' < \kappa$ be such that $\alpha \le \alpha'$ and $\bigwedge_{\ell<k} u_\ell \subseteq \alpha'$. Let D' be the filter generated by

$$D \cup \left\{ \left\{ n \in \bigcap_{\ell<k} \operatorname{dom}(f_{j_\ell}) : f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \ne \emptyset \right\} : \right.$$
$$\left. k < \omega, j_\ell < i, u_\ell \in [\alpha' \setminus \alpha]^{<\aleph_0}, \eta_\ell \in {}^{u_\ell}2 \right\}.$$

Then $\omega \setminus B \in D'$, and $(D', i, \alpha') \in \mathcal{AP}$.

$\boxtimes_4$ If $(D, i, \alpha) \in \mathcal{AP}$ then for some D', α' we have that $(D', i+1, \alpha') \in \mathcal{AP}$.

Proof. Let $M \prec (H(\chi), \in)$ such that $M \cap \kappa \in \kappa$, $(D, i, \alpha) \in M$, $\mathcal{F}_g \in M$, and $|M| < \kappa$. Suppose that $\operatorname{dom}(f_i) \in D$ and that $\lim_D \langle f^1_i(k) : k \in \operatorname{dom}(f_i)\rangle = \omega$. Let $\alpha' = M \cap \kappa$. Let D_1 be the filter in the boolean

algebra in $\mathcal{P}(\omega) \cap M$ generated by

$$(D \cap M) \cup \left\{ \left\{ n \in \bigcap_{\ell<k} \mathrm{dom}(f_{j_\ell}) \,:\, f^2_{j_\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset \right\} \,: \right.$$
$$\left. k < \omega, j_\ell \leq i, u_\ell \in [\alpha' \setminus \alpha]^{<\aleph_0}, \eta_\ell \in {}^{u_\ell}2 \right\}.$$

Since in M, $\bar{A}$ is (g,κ)-o.k., this has the infinite intersection property. Let D'_2 be an ultrafilter in M extending D_1. Let D' be the filter on ω in V that D'_2 generates.

Now we take a maximal element in the partial order $(\mathcal{AP}, \leq_{\mathcal{AP}})$. By the properties $\boxtimes_1$ to $\boxtimes_4$ it is as required in $(*)$. $\square$

Note that $(*)$ of 3.7 implies that $\bar{A}$ is (g,κ)-o.k. The following is a preservation theorem for suitable ultrafilters:

Claim 3.8. *Assume that*

(a) $\bar{A}$ *is* (g,κ)*-o.k.*

(b) $D = \langle D_\eta \,:\, \eta \in {}^{<\omega}\omega \rangle$, $D_\eta = D$, D *is ultrafilter on* ω *as in 3.7.*

(c) $Q_D = \{T \,:\, T \subseteq {}^{<\omega}\omega$ *is a subtree, and for some* $\eta \in T, \eta \trianglelefteq \nu \in T \Rightarrow \{k \,:\, \nu\hat{\ }k \in T\} \in D_\nu\}$, *ordered by inverse inclusion. (The* $\triangleleft$*-minimal* η *of this sort is called the trunk of* T, $\mathrm{tr}(T)$*.)*

Then $\Vdash_{Q_D}$ *"*$\bar{A}$ *is* (g,κ)*-o.k.".*

Proof. We use the fact [8] that Q_D has the pure decision property: Let φ_i, $i \in \omega$, be countably many sentences of the Q_D-forcing language. We think of names $\underset{\sim}{f}_\ell$, $\ell < k$, for some elements of $\mathcal{F}_g$ and $\varphi_i =$ "$\Big($the i-th element of $\underset{\sim}{B} = \bigcap_{\ell<k} \mathrm{dom}(\underset{\sim}{f}_\ell)\Big) = \check{b}_i$ and $\bigwedge_{\ell<k} \underset{\sim}{f}_\ell(\check{b}_i) = (\check{f}^1_{\ell,i}, \check{f}^2_{\ell,i})$". The pure decision property says:

$$\forall p \in Q_D \, \exists q \geq_{tr} p \, \forall r \geq q \, \forall i \left(r \Vdash \varphi_i \rightarrow (\exists s_i \in r) q^{[s_i]} \Vdash \varphi_i \right),$$

where we write $\geq_{tr}$ for the pure extension: $q \leq_{tr} r$ if $r \subseteq q$ and $\mathrm{tr}(q) = \mathrm{tr}(r)$, and $q^{[s_i]} = \{\eta \in q \,:\, s_i \trianglelefteq \eta\}$.

Towards a contradiction we assume that there is a counterexample. By Claim 3.3 (first (b) and then (a)) we may assume that it is of the following

form

$$p^* \Vdash \text{“}\langle \underset{\sim}{f}_\ell : \ell < k\rangle \text{ form a task}$$

$$\text{such that the intersection of the domains is } B = \omega$$

$$(**)\qquad \text{and for } i \in B,\ \max\{\underset{\sim}{f}^1_\ell(i) : \ell < k\} < \min\{\underset{\sim}{f}^1_\ell(i+1) : \ell < k\}$$

$$\text{and there is no } \alpha < \kappa \text{ such that the statement}$$

$$\text{(3.1) from Definition 3.2(3)(c) holds.”}$$

We find q such that

(α) $q \in P$

(β) $q \geq_{tr} p^*$,

(γ) for all $i \in \omega$ for all $f^1_{\ell,i} \in \omega$, $f^2_{\ell,i} \subseteq [0, g(f^1_{\ell,i}))$ of size bigger than $f^1_{\ell,i}$ we have that

if $r \geq q, r \Vdash$ “$\underset{\sim}{f}_\ell(\check{i}) = (\check{f}^1_{\ell,i}, \check{f}^2_{\ell,i})$”,

then also for some $s_i \in r$, the condition $q^{[s_i]}$ forces the same.”

We fix such a q.

Now we set for $\nu \in q$ and $\ell < k$

$$B^1_{\nu,\ell} = \{i \in \omega : \text{ some pure extension of } q^{[\nu]} \text{ decides } \underset{\sim}{f}_\ell(i)\}.$$

We say (ν, ℓ) is 1-good if $B^1_{\nu,\ell} \in D$. Let for $i \in B^1_{\nu,\ell}$, $h_{\nu,\ell}(i) = (h^1_{\nu\ell}, h^2_{\nu,\ell})$ the value of $\underset{\sim}{f}_\ell(i)$ that is given by the pure decision. This is well-defined because any two pure extensions are compatible. Of course, by the requirements we had put on the counterexample, we have that $\lim_D \langle h^1_{\nu,\ell}(i) : i \in B^1_{\nu,\ell}\rangle = \omega$.

We say that $(\nu, \ell) \in q \times k$ is 2-good, if it is not 1-good and we have for all $m \in \omega$ that

$$M_{\nu,\ell,m} = \{j \in \omega : (\exists i \in \omega)(h_{\nu\hat{}j,\ell}(i)) \text{ is well-defined,}$$
$$\text{and } h^1_{\nu\hat{}j,\ell}(i) > m)\big\} \in D.$$

So, for 2-good but not 1-good (ν, ℓ) we may define for $j \in M_{\nu,\ell,m}$,

$$g_{\nu,\ell}(j) = h_{\nu\hat{}j,\ell}(i_{\nu\hat{}j,\ell}),$$

where $i_{\nu\hat{}j,\ell}$ is such that $h_{\nu\hat{}j,\ell}(i_{\nu\hat{}j,\ell})$ is defined in $h^1_{\nu\hat{}j,\ell}(i_{\nu\hat{}j,\ell}) > m$

and if there is a maximal such i, then take this as $i_{\nu\hat{}j,\ell}$.

We show that there is $M'_{\nu,\ell,m} \in D$, $M_{\nu,\ell,m} \supseteq M'_{\nu,\ell,m}$ such that for $j \in M'_{\nu,\ell,m}$ there a maximal such i: If $h_{\nu\hat{}j,\ell}(i)$ is defined and $i' < i$ then there is some pure extension deciding $h_{\nu\hat{}j,\ell}(i')$ since there are only finitely many possibilities for it values, by the third line of $(**)$. Hence some pure extension decides the value.

Hence also $h_{\nu\hat{}j,\ell}(i')$ is defined. If $h_{\nu\hat{}j,\ell}(i)$ is defined for all i, then $(\nu\hat{}j,\ell)$ is 1-good. Hence , if $(\nu\hat{}j,\ell)$ is 2-good but not 1-good, then there is a maximal i witnessing $j \in M_{\nu,\ell,m}$. If $\{j : (\nu\hat{}j,\ell) \text{ is 1-good }\} \in D$, then by gluing together suitable pure extensions r_j of $q^{[\nu\hat{}j]}$ together we get a pure extension of $q^{[\nu]}$ that shows that (ν,ℓ) is 1-good. Hence $X = \{j : (\nu\hat{}j \text{ is 2-good and not 1-good }\} \in D$. So we may take $M'_{\nu,\ell,m} = M_{\nu,\ell,m} \cap X$. In order to simpily notation, we assume that $M'_{\nu,\ell,m} = M_{\nu,\ell,m}$.

Also from the third line of $(**)$ we get that for every $\nu \in q$ either for all $\ell < k$, (ν,ℓ) is 1-good or no (ν,ℓ) is 1-good. In the latter case there is some i_ν, such that for all $\ell < k$, $\mathrm{dom}(h_{\nu,\ell}) = i_\nu$ or $\mathrm{dom}(h_{\nu,\ell}) = i_\nu + 1$. Moreover, also by $(**)$ we get that if for some $\ell < k$, for all m, $M_{\nu\ell,m} \in D$, then for all $\ell < k$, for all m, $M_{\nu,\ell,m} \in D$. So if (ν,ℓ) is 2-good, then all (ν,ℓ') are 2-good. We call ν i-good if there is some ℓ such that (ν,ℓ) is i-good. We set $M_{\nu,m} = \bigcap_{\ell<k} M_{\nu,\ell,m}$.

We fix some diagonal intersection M_ν of $\langle M_{\nu,m} : m \in \omega\rangle$, such that $\lim\langle i_{\nu\hat{}j} : j \in M_\nu\rangle = \omega$.

Then we also have that $\lim_D\langle \min\{g^1_{\nu,\ell}(j) : \ell < k\} : j \in M_\nu\rangle = \omega$, because for each $z < \omega$, $\{j : \min\{g^1_{\nu,\ell}(j) : \ell < k\} < z\}$ is a cofinite set. Hence $g_{\nu,\ell} \in \mathcal{F}_g$. By combining with an enumeration of M_ν, we may assume that $\mathrm{dom}(g_{\nu,\ell}) = \omega \in D$. We will not write this enumeration, in order to prevent too clumsy notation, but we shall later apply that D is as in 3.7 for $\mathcal{F}_g$, and therefore we need that the domains are in D.

Now we take χ sufficiently large and $N \prec (H(\chi),\in)$ such that $\langle \underset{\sim}{f}_\ell : \ell < k\rangle \in N$, $\langle B^1_{\nu,\ell}, h_{\nu,\ell}, g_{\nu,\ell} : \nu \in q, \ell < k\rangle \in N$, $q, D \in N$. We take $\alpha^* = \sup(N \cap \kappa)$. We claim that q forces that α^* is as in the Definition 3.2(3)(c).

If not, then there are counterexamples $u_\ell \in [\kappa \setminus \alpha^*]^{<\aleph_0}$ and $\eta_\ell \in {}^{u_\ell}2$ and $r \in Q_D$, $r \geq q$, and b^* such that

$$(\diamond\diamond)\qquad \begin{aligned} & r \geq q, \text{ and} \\ & r \Vdash_{Q_D} \text{“} \bigcap_{\ell<k} \mathrm{dom}(\underset{\sim}{f}_\ell) = \omega \text{ and} \\ & \quad (\forall i \in \omega) \max\{\underset{\sim}{f}^1_\ell(i) : \ell < k\} < \min\{\underset{\sim}{f}^1_\ell(i+1) : \ell < k\} \\ & \quad \text{and } \left\{ b \in \omega : (\forall \ell < k)(\underset{\sim}{f}^2_\ell(b) \cap \underset{\sim}{\bar{A}}^{[\eta_\ell]} \neq \emptyset) \right\} \subseteq [0, b^*]\text{”}. \end{aligned}$$

First case: There is some $\nu \in r$ with $\mathrm{tr}(r) \trianglelefteq \nu$ such that all ν is 1-good. Now we take for each $t \in \omega$, some pure extension of $q_t^{[\nu]}$ of $r^{[\nu]}$ such that it forces $\bigwedge_{\ell<k}(h_{\nu,\ell} \restriction t = \underset{\sim}{f}_\ell \restriction t)$. Since $\bar{A}$ is (g,κ)-o.k., and since all is reflected to N and by the choice of α^* we have that $I = \{n \in \omega : (\forall \ell < k)(h^2_{\nu,\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite. So we take $t \in I$ such that $t > b^*$. Now $q_t^{[\nu]}$ contradicts $(\diamond\diamond)$.

Second case. There is some $\nu \in r$ such that all ν, $\ell < k$ are 2-good but not 1-good. We set $g_{\nu,\ell}(j) = h_{\nu\hat{}j,\ell}(i_{\nu\hat{}j,\ell})$ as purely decided above $q^{[\nu\hat{}j]}$. Fact: Now $\langle g_{\nu,\ell} \,:\, \ell < k\rangle$ is as required in the definition of $\bar{A}$ being (g,κ)-o.k., because $\omega = \lim_D \langle g^1(i_{\nu\hat{}j}) \,:\, j \in \omega\rangle$.

Now we take for each $t \in \omega$, some pure extension of $q_t^{[\nu\hat{}j]}$ of $r^{[\nu\hat{}j]}$ such that it determines $\bigwedge_{\ell<k} g_{\nu,\ell} \restriction t$. Since $\bar{A}$ is (g,κ)-o.k., and since all is reflected to N and by the choice of α^* we have that $J = \{n \in \omega \,:\, (\forall \ell < k)(g^2_{\nu,\ell}(n) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite. Then also $\hat{J} = \{i_{\nu\hat{}n} \,:\, n \in J\}$ is infinite. So we take $t > b^*$, $t \in \hat{J}$. Now the gluing together of $q_t^{[\nu\hat{}j]}$, $j \in \bigcap_{\ell<k} M_{\nu,\ell,t}$, contradicts $(\diamond\diamond)$ because we have $g_{\nu,\ell}(j) = h_{\nu\hat{}j,\ell}(i_{\nu\hat{}j,\ell}) = f_\ell(i_{\nu\hat{}j})$, if $q_t^{[\nu]} \in G$. Here we write f_ℓ for $\underset{\sim}{f}_\ell[G]$.

Third case: All $\nu \in r$ are neither 1-good nor 2-good. We shall prove something stronger:

> An end-segment of the generic $\bigcup\{\eta \,:\,$ there is some element $q \in G$ with trunk $\eta\}$ can be thinned out (such that still infinitely many points are left) and injected into an infinite subset of $\{n \in \omega \,:\, \bigwedge_{\ell<k} \underset{\sim}{f}^2_\ell[G](n) \cap A^{[\eta_\ell]} \neq \emptyset\}$.

This is more than enough.

Let $i_{\nu,\ell} = \max(B^1_{\nu,\ell}) < \omega$, because ν is not 1-good. Let $i^*_\nu = \mathrm{dom}(h_{\nu,\ell})$ such that $i^*_\nu = i^*_{\nu,\ell}$ or $i^*_\nu = i^*_{\nu,\ell} + 1$. By the premise $(**)$, there are such i^*_ν. There is $r \geq q$ with no $\nu \in r$ being 1-good or 2-good in N. W.l.o.g. we take q like that. Now we try to shrink q purely. Let $\nu_0 = \mathrm{tr}(q)$.

First: We have that $\underset{\sim}{f}_\ell \restriction i^*_\nu$ is decided by q. The range of $\langle i^*_{\nu\hat{}j} \,:\, \nu\hat{}j \in q\rangle$ is bounded modulo D because ν is not 2-good. Hence we may assume that there is just one value i^{**}_ν. So say (after shrinking q) that it is constant with value $i^{**}_\nu \geq i^*_\nu$.

Second we have that $\nu_0 \trianglelefteq \nu \in q$ implies that $q^{[\nu]}$ decides $\underset{\sim}{f}_\ell \restriction i^{**}_\nu$.

Third we have that if $i \in [i^*_\nu, i^{**}_\nu]$ then $\lim_D \langle \underset{\sim}{f}^1_{\nu\hat{}j,\ell}(i) \,:\, j \in \omega\rangle = \omega$ by the definition of i^*_ν and i^{**}_ν. So define $g_{\nu,\ell,i}$ by $g_{\nu,\ell,i}(j) = h_{\nu\hat{}j,\ell}(i)$. So $g_{\nu,\ell,i} \in N$ is a function of the right form.

We have by the definition of α^* that for all $i \in [i^*_\nu, i^{**}_\nu)$ for all $\nu \in q$ for all u_ℓ, η_ℓ that

$$A := \{b \,:\, (\forall \ell < k) g^2_{\nu,\ell,i}(b) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\} \in D.$$

Since the range of $\bigcup\{\eta \,:\,$ there is some element $q \in G$ with trunk $\eta\} =: \eta_\omega$ is eventually contained is every set in D, we now find the following infinite set: We take $\langle \eta_n \,:\, n \in \omega\rangle$ such that $\eta_n \in \mathrm{range}(\eta_\omega) \cap A$ and such that $i^{**}_{\eta_n} < i^*_{\eta_{n+1}}$. We set $\xi_n = \eta_n \restriction |\eta_n - 1|$. Then we have for almost all n such that $\xi_n \in A$ and hence

for all $i \in [i^*_{\xi_n}, i^{**}_{\xi_n})$: $g_{\xi_n,\ell,i}(\eta_n(|\eta_n - 1|)) = h_{\xi_n\hat{}\eta_n(|\eta_n-1|),\ell}(i) = h_{\eta_n,\ell}(i) = f_\ell(i)$. So $\bigcup_{n\in\omega}[i^*_{\xi_n}, i^{**}_{\xi_n}] \subseteq^* \{b : (\forall \ell < k) f^2_\ell(b) \cap \bar{A}^{[\eta_\ell]} \neq \emptyset\}$ is infinite.

□

Claim 3.9. *Let $\kappa = \mathrm{cf}(\kappa) > \omega_1$. Let $\mathbf{V}_0 \models$ CH and let $P_0 = \mathbb{C}_\kappa$ be the forcing adding κ Cohen reals. We fix some function $g \in \mathbf{V}_0$, so that every hyperarithmetic function in $\mathbf{V}_0$ is computable in every $g' \geq g$. Set $\mathbf{V}_1 = \mathbf{V}_0[G_0]$. Let in $\mathbf{V}_1$, $\bar{A}$ be the enumeration of the κ Cohen reals.*

(1) *In $\mathbf{V}_1$, there is $(P, \bar{A}) \in K_g$ such that $\Vdash_P$ "$\mathfrak{r} < \kappa$", even $\Vdash_P$ "$\mathfrak{r} = \aleph_1$"*

(2) *For $(P, \bar{A})$ as in (1), we have that in $\mathbf{V}_1$, $\Vdash_P$ "every hyperarithmetic real is weakly needed for the reaping relation".*

Proof. (1) By 3.5 we have that $\bar{A}$ is (g, κ)-o.k. in $\mathbf{V}_1$. According to 3.7, we may choose in $\mathbf{V}_1$ $\lessdot$-increasing and continuous such that $(P_i, \bar{A}) \in K$, $P_{i+1} = P_i * Q_{\underset{\sim}{D}^i}$, where $\underset{\sim}{D}^i = \langle D^i_\eta : \eta \in {}^{<\omega}\omega\rangle$ $D^i_\eta = D^i \in V^{P_i}$ as in 3.7. Note that $P = \bigcup_{i<\omega_1} P_i$ forces that $\mathfrak{r} = \aleph_1$, because it consecutively adds ("shoots") $\aleph_1$ reals through ultrafilters in the intermediate models $\mathbf{V}_0[G_\alpha]$, $\alpha < \omega_1$. It is easy to see that these $\aleph_1$ reals are a refining family.

(2) Now by part (1) and by 3.4 for any g the proof of (2) follows.

4. There may be more weakly needed reals than needed reals

Under CH, or if $||R|| = 2^{\aleph_0}$, then needed for R and weakly needed for R coincide. In this section, we show that there is some quite simply defined relation R and that there is some model of ZFC in which there are more weakly needed reals for R than needed reals for R. The idea is to use the forcing model from the previous section.

Claim 4.1. *(Blass [3]) An equivalent condition for "$\eta \in {}^\omega 2$ is needed for R" is*

$$(\exists x \in \mathrm{dom}(R))(\forall y \in \mathrm{range}(R))(xRy \rightarrow \eta \leq_T y).$$

(2) *If $2^{\aleph_0} = \aleph_1$ then "needed for R" is equivalent to "weakly needed for R" (and for the usual R's, under MA we have that $||R|| = 2^{\aleph_0}$ and hence any adequate set is of minimal cardinality and hence the notions coincide).*

Proof. Suppose that η is needed for R and that there is no x as in (1). Then $(\forall x \in \mathrm{dom}(R))$ $(\exists y \in \mathrm{range}(R))(xRy \wedge \eta \not\leq_T y)$. So we can build a R-adequate set from all these y's, that shows that η is not needed for R. For the other implication: Fix x as in (1). Every R-adequate set has to contain one y such that xRy and hence $\eta \leq_T y$. □

If $2^{\aleph_0} = \aleph_1$ then "needed for R" is equivalent to "weakly needed for R" (and for the usual R's, under MA we have that $||R|| = 2^{\aleph_0}$ and hence any adequate set is of minimal cardinality and hence the notions coincide). But in general, they do not coincide.

Claim 4.2. *There is a simply defined relation R for which it is consistent that the notions "weakly needed" and "needed" do not coincide. In fact, in the forcing model from the previous section, every R-needed real is recursive, and all the hyperarithmetic (and possibly more) reals are weakly needed for R.*

Proof. Let $R = R_0 \cup R_1$, where R_0 is the ordinary reaping relation, which we write for functions on ${}^\omega 2 \times {}^\omega 2$:

$$\eta R_0 \nu \Leftrightarrow \eta, \nu \in {}^\omega 2 \ \wedge \ (\exists^\infty n)\nu(n) = 1 \ \wedge \ \eta \restriction \nu^{-1}\{1\} \text{ is almost constant.}$$

$$\eta R_1 \nu \Leftrightarrow \eta, \nu \in {}^\omega 2 \ \wedge \ (\exists^\infty n)\eta(n) = 1 \ \wedge \ (\exists^\infty n)\nu(n) = 1 \ \wedge$$

$$\left\langle \frac{|\nu^{-1}\{1\} \cap \eta^{-1}\{1\} \cap n|}{|\eta^{-1}\{1\} \cap n|} \, : \, n \in \omega \right\rangle \text{ converges to } \frac{1}{2}.$$

$$\text{In particular, for every large enough } n, \ \frac{|\nu^{-1}\{1\} \cap \eta^{-1}\{1\} \cap n|}{|\eta^{-1}\{1\} \cap n|} \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

We use V^P from the previous section. There we have that $P = P_0 * \underset{\sim}{Q}$, P_0 is the forcing adding κ Cohen reals, and $\underset{\sim}{\bar{A}}$ is an enumeration of the names of these Cohen reals, and Q is the iteration described in 3.8. Then in V^P we have that $||R|| \leq ||R_0|| = \aleph_1$.

We first show that every hyperarithmetic real is weakly needed for R in this model. We take some R-adequate set in V^P $\mathcal{R}$ of power $\aleph_1$. We let

$$Y_\ell = \{i < \kappa \, : \, (\exists x \in \mathcal{R})(A_i R_\ell x)\}.$$

So, by the definition of adequate we have that $Y_0 \cup Y_1 = \kappa$. If $|Y_0| = \kappa$, by the proof of 3.4, we get some $x \in \mathcal{R}$ whose enumeration f with $f(n) = m$ if m is the nth element of x is so large in the eventual domination order that hyperarithmetic real is computable from it.

We now show that $|Y_1| < \kappa$. Then it follows that $|Y_0| = \kappa$. Towards a contradiction, we assume that $|Y_1| = \kappa$. In the model from the previous section we have that $P = \bigcup_{i<\omega_1} P_i$, P_0 adds κ Cohen reals, P_i increasing and continuous, $P_{i+1} = P_i * Q_{\underset{\sim}{D}_i}$ as there, $P = P_0 * \underset{\sim}{Q}$. We work in V^{P_0}. We have that for some $p^* \in Q/P_0$ and some Q/P_0-names $\underset{\sim}{\nu}_i$, $i < \omega_1$

$$p^* \Vdash_{Q/P_0} |\underset{\sim}{Y}_1| = \kappa \ \wedge \ \underset{\sim}{\mathcal{R}} = \{\underset{\sim}{\nu}_i \, : \, i < \omega_1\}.$$

$Y^* = \{\alpha \, : \, \exists p_\alpha \geq p^*, p_\alpha \Vdash_{Q/P_0} \alpha \in \underset{\sim}{Y}_1\}$. By the ccc of Q/P_0, we have that $Y^* \in [\kappa]^\kappa$, and for $\alpha \in Y^*$ we choose $p^* \leq p_\alpha \Vdash_{Q/P_0}$ "$\alpha \in \underset{\sim}{Y}_1$" So for $\alpha \in \kappa$

we have that $A_\alpha R_1 \nu_{i(\alpha)}$ and hence for a large enough n^* for κ many $\alpha \in Y^*$ (w.l.o.g.: for all $\alpha \in Y^*$) we have that $n_\alpha = n^*$, and there is a Δ-system for the $\operatorname{dom} p_\alpha \in [\kappa \setminus \{0\}]^{<\omega}$ whose root is u^*, $i(\alpha) = i^*$.

So we may assume that for $j \in u^*$ we have that $p_\alpha(j)$ is an object with trunk ρ_j and not just a P_0-name. By pure decidability for some $\nu^* \in V^{P_0}$ we have: For every $\alpha \in Y^*$ and m for some pure extension q of p_α with the same domain $q \Vdash \underset{\sim}{\nu}_{i(*)} \restriction m = \nu^* \restriction m$. By "$n_\alpha = n(*)$" for $\alpha \in Y^*$ we get an easy contradiction: Suppose $p \in P_0$ and

$$p \Vdash_{P_0} \text{“}\forall \alpha \in Y^* \, \forall n \geq n(*) \, \exists q_\alpha \geq_{tr} p_\alpha,$$
$$q_\alpha \Vdash_{Q/P_0} \text{“}\frac{|\underset{\sim}{\nu}^{*-1}\{1\} \cap A_\alpha^{-1}\{1\} \cap n(*)|}{|A_\alpha^{-1}\{1\} \cap n(*)|} \in \left[\frac{1}{4}, \frac{3}{4}\right]\text{” ”}.$$

This is impossible, because we may assume that $\nu^* \in V$ (it needs only countably many of the κ Cohen reals) and we may arrange all other A_α's so that the quotient will be arbitrary. The forcing P/P_0 does not change this fact.

Now we show that if a real is not recursive then it is not needed for R. If η is not recursive and $x \in {}^\omega 2$, let $\{x, \eta\} \in N \prec (H(\chi), \in)$, N countable. Let $\nu = \nu(x, \eta)$ be random over N, and we claim

$$\eta \not\leq_{Turing} \nu. \tag{4.1}$$

Proof of (4.1): Otherwise we would have that η is recursive in the ground model by the following: Suppose

$$p \Vdash_{\text{Random}} \text{“}M \text{ computes } \eta \text{ from the oracle } \underset{\sim}{\nu}\text{”}. \tag{4.2}$$

Then by the Lebesgue density theorem we find $s \in {}^{<\omega}2$ such that above s, p has Lebesgue measure $> \frac{99}{100} \cdot \operatorname{Leb}(\{\rho \,:\, s \triangleleft \rho\}$. The we set

$$B_n = \{\nu' \in {}^\omega 2 \,:\, s \triangleleft \nu' \text{ and from } \nu' \; M \text{ computes } \eta(n) \text{ correctly}\}.$$

From (4.2) we get that $\operatorname{Leb}(B_n) \geq \frac{99}{100} \cdot \operatorname{Leb}(\{\rho \,:\, s \triangleleft \rho\}$. So for every sufficiently large $m \in \omega$ we have that

$$2^{m - \lg(s)} \leq |\{\nu' \in {}^m 2 \,:\, s \triangleleft \nu' \text{ and from } \nu' \; M \text{ computes } \eta(n) \text{ correctly}\}|. \tag{4.3}$$

So we can run a machine, that has s as an fixed ingredient, and which, given input n, increases m successively, and then computes $\eta(n)$ with all possible oracles above s of length $m \geq \lg(s)$ and decides with (4.3), when it is true for m (and hence for all later m), which is the right value. So (4.1) is proved.

But we have that $x R_1 \nu$ and hence $x R \nu$. Thus the collection $\{\nu(x, \eta) \,:\, x \in {}^\omega 2\}$ is an R-adequate family. So there is some ν such that $\eta \leq_T \nu$ in contradiction to the equation (4.1). So finally we showed that all needed reals for R are

recursive. □

5. Needed reals for reaping

In this section we prove in ZFC that not all hyperarithmetic reals are needed for the reaping relation. Since in the model from Section 3 all hyperarithmetic reals are weakly needed for the reaping relation, this model shows that also for the reaping relation it is consistent that weakly needed and needed do not coincide.

Hypothesis 5.1. *We fix $B^* \subseteq \omega$ and some $\eta \in {}^{\omega}2$ such that: if $X \subseteq B_* = B^*_1$ or $X \subseteq \omega \setminus B_* = B^*_2$ then η is recursive in ch_X.*

By 4.1, the hypothesis says, that η is needed for the reaping relation, with witness B^*. For all X, that refine B^*, we have that η is recursive in X. Note that 5.1 is similar to η being hyperarithmetic: the difference is that η is computable also in every infinite subset of the complement of B_*.

Choice 5.2. *Let $\langle (M^n_1, M^n_2, a^n_1, a^n_2) : n < \omega \rangle$ be a recursive list of the quadruples (M_1, M_2, a_1, a_2) such that*

- *(i) M_1, M_2 are Turing machines (with reference to an oracle),*
- *(2) a_1, a_2 are finite disjoint sets.*

W.l.o.g. $a^n_1 \cup a^n_2 \subseteq n$ and each quadruple appears infinitely often.

Definition 5.3. *(1) We say $\bar{E} = \langle E_n : n \in \omega \rangle$ is special if*

- *(i) E_n is an equivalence relation on $\omega \setminus n$, and*
- *(ii) for $m < n$, E_n refines $E_m \restriction (\omega \setminus n)$,*
- *(iii) if A is an E_n-equivalence class, then $A \setminus (n + 1)$ is devided by E_{n+1} in at most two equivalence classes, and E_0 has finitely many classes,*
- *(iv) if*
  - *(α) A is an E_n-equivalence class and*
  - *(β) there is a partition X_1, X_2 of $A \setminus (n+1)$ such that for all $j < \omega$, $Y_i \subseteq \omega$, $i = 1, 2$, (if $a^n_i \subseteq Y_i \subseteq X_i \cup a^n_i$, $h_i < \omega$, the machine M^n_i running with input j and oracle ch_{Y_i} finishes its run giving h_i, then $h_1 = h_2$),*

  then E_{n+1} induces such a partition of A.

(2) $\bar{E}$ is special to η if in addition

(v) *for all A and n, if A is an E_n-class, then η is not recursive in ch_A.*

Theorem 5.4. *There is no $\bar{E}$ that is special to η.*

Proof. We assume the contrary, and by (Cohen) forcing and absoluteness we will derive a contradiction. The proof will be finished with 5.11.

Definition 5.5. *For a special $\bar{E}$ we define $Q = Q_{\bar{E},B_*}$ as the following notion of forcing:*

(1) *$p \in Q$ has the form $p = (n, A, b_1, b_2) = (n^p, A^p, b_1^p, b_2^p)$ such that*
   (i) $n < \omega$,
   (ii) *A is an E_n-equivalence class,*
   (iii) *A is infinite,*
   (iv) *b_1, b_2 are disjoint subsets of n,*
   (v) $b_1 \subseteq B^*$, $b_2 \subseteq \omega \setminus B^*$.

(2) *$p \leq q$ iff*
   (i) $n^p \leq n^q$, $A^p \supseteq A^q$, $b_i^p \subseteq b_i^q$, *for* $i = 1, 2$,
   (ii) $(b_1^q \cup b_2^q) \setminus (b_1^p \cup b_2^p) \subseteq A^p$.

(3) *$\underset{\sim}{B}_i = \bigcup\{b_i^p : p \in \underset{\sim}{G}_Q\}$ is a Q-name of a subset $B_i \in V[G]$ of B_i^* if $i = 1, 2$.*

So if E_0 has finitely many equivalence classes, then Q is equivalent to Cohen forcing and independent of $\bar{E}$ and B_*. Nevertheless we keep the complicated conditions, because they are better tailored for η's needed for the reaping relation.

Claim 5.6. *For $i = 1, 2$ we have*

(1) *$\Vdash_Q$ "$\underset{\sim}{b}_i$ is an infinite subset of B_i^*".*

(2) *For some p^*, $p^* \Vdash_Q$ "$M_i^{n^{p^*}}$ computes η with the oracle $ch_{\underset{\sim}{b}_i}$".*

Proof. (1) Fix $i = 1$ or $i = 2$. It is enough, to find for a given $p \in Q$ some $q \geq p$, $q \in Q$ such that $b_i^p \subsetneqq b_i^q$. Now $A^p \cap B_i^*$ is infinite, because of the hypothesis on B^* and because η is not recursive in ch_{A^p} by the assumption of the indirect proof of 5.4. We may choose $h \in A^p \cap B_i^*$, $h \geq \max(b_i^p)+1$ and an infinite E_{h+1}-class $A \subseteq A^p$, which exists because A^p is infinite and because E_{h+1} has finitely many equivalence classes. We define q as $n^q = h + 1$, $A^q = A$, $b_i^q = b_i^p \cup \{h\}$, $b_{3-i}^q = b_{3-i}^p$.

(2) The statement made in Hypothesis 5.1 on B^* and on η is Π^1_1 and holds in V, hence it holds in $V[G]$ as well by [5, Theorem 98, p. 530]. Now we apply it in $V[G]$ to part (1) of this claim. $\square$

We fix p^*, $M_1^{n^{p^*}}$, $M_2^{n^{p^*}}$ as in part (2) of Claim 5.6.

Fact 5.7. *There is some $q \geq p^*$ such that for $i = 1, 2$, $M_i^{n^q} = M_i^{n^{p^*}}$ and such that $b_i^q = a_i^{n^q}$.*

Proof. For some $n^* \geq n^{p^*}$ the quadruple $(M_1^{n^*}, M_2^{n*}, a_1^{n^*}, a_2^{n^*})$ is equal to $(M_1^{n^{p^*}}, M_2^{n^{p^*}}, b_1^{p^*}, b_2^{p^*})$. Let A be an infinite E_{n^*}-class which is a subset of A^{p^*}. So we take $q = (n^*, A, a_1^{n^*}, a_2^{n^*})$.

Claim 5.8. *For n^*, A the demands $(\alpha) + (\beta)$ of clause (iv) of 5.3 hold, hence the conclusion.*

Proof. We work first in $V[G]$. There, by 5.6, $X_i = A \cap B_i^*$ and A exemplify 5.3(iv). But 5.3(iv) is a Σ^1_2-statement of the parameters (A, a_1^n, a_2^n), and therefore it holds in V as well by Shoenfield's absoluteness theorem [5, Theorem 98, p. 530]. $\square$

Convention 5.9. *Let $A_1 \neq A_2$ be the E_{n^*+1}-equivalence classes which are subsets of A, with A_i for M_i as in 5.3(iv).*

Claim 5.10. *If $j < \omega$ then for some $b \subseteq m < \omega$ we have that $b \cap n^q = b_1^q$, $b \setminus n^q \subseteq A_1$, if we let M_1 run with input j and oracle $ch_b \restriction m$ it gives an answer (i.e. it finishes and asks the oracle only questions in its domain m).*

Proof. : We define $r \in Q$ by $n^r = n^q + 1$, $A^r = A_1$, $b_i^r = b_i^q$ for $i = 1, 2$. So $q \leq r \in Q$. By the choice of q for some $s \in Q$, $r \leq s$ and s forces a value to the run of M_1 with input j and oracle $\underset{\sim}{b}_1$ so also to the answers to the oracle in this run. Let $b = b_1^s$. $\square$

Claim 5.11. *For every $j \in \omega$, $k \in 2$ the following are equivalent*

(1) $\eta(j) = k$.

(2) *For some $b \subseteq m < \omega$, $b \cap n^q = b_1^q$, $b \setminus n^{p^*} \subseteq A_1$, and M_1 running with input j and oracle $ch_b \restriction m$ gives the answer k.*

Proof. $(i) \to (ii)$ by 5.10. Since $\eta \in V$, the reverse implication holds as well. $\square$

End of the proof of 5.4: η is recursive in ch_{A_1}. By 5.11 we try all b's for a given j and hence η is recursive in ch_{A_1}. How to run through all trials is explained in more detail in [4, Theorem 9]. $\square_{5.4}$

Claim 5.12. *There is a special $\bar{E}$ that has as a three place relation $\{\langle n, x, y\rangle : xE_ny\}$ Turing degree $\leq O^\omega$ and such that if A is an E_n-equivalence class then $ch_A \leq_T O^{n+1}$, the $(n+1)$st jump of O.*

Proof. We choose E_n by induction on n.

$n = 0$. If for every m there is a partition (c_0, c_1) of m such that for $i \in \{1, 2\}$ for every $b_i \subseteq c_i$ and $j < n$ and M_i^0 running with input j and oracle $\mathrm{ch}_{b_i} \restriction m$ or $\mathrm{ch}_{b_i} \restriction m$ and giving the results k_i then $k_0 = k_1$, then we choose among these pairs (c_1^m, c_2^m) such that $\mathrm{ch}_{c_1^m}$ is minimal in the lexicographical order. If (c_1^m, c_2^m) are defined for every m, then we have that $m^1 \leq m^2 \leq m^3 \Rightarrow \mathrm{ch}_{c_1^{m_2} \cap m_1} \leq_{lex} \mathrm{ch}_{c_1^{m_3} \cap m_1}$. So $\langle c_1^m : m \in \omega\rangle$ converges to some c_1. Now we define E_0, having two classes: c_1 and $\omega \setminus c_1$. The relation E_0 is computable in O^1.

In the step from n to $n+1$, the relation E_{n+1} is defined similarly, with the modification that we use the description of E_n as a parameter and take partitions (c_0, c_1) of $(m \setminus n) \cap C$ for each E_n-class C, and oracles $B_i \cup a_i^n$. Clearly using O^{n+2} we can choose E_{n+1} and $\bar{E}$ is $\ll \Delta_1^1$. $\square$

Remark 1. Just to show that Con(needed for reaping does not coincide with weakly needed for reaping) is is enough to find a Δ_1^1-relation $\bar{E}$ which is special.

Conclusion 5.13. *If η is needed for the reaping relation, then $\bigvee_{n\in\omega}(\eta \leq_T O^n)$, hence in the V^P from Section 3 many Δ_1^1 reals are not needed for the reaping relation, but only weakly needed for the reaping relation.*

Proof. We take $\bar{E}$ as in 5.12. From 5.11 we get that $\bar{E}$ is not special to any η that satisfies 5.1 for some B_*. So any η that is needed for the reaping relation is recursive in $\bar{E}$. $\square$

6. Coincidence

In this section we give a condition on a relation R under which needed for R and weakly needed for R coincide and show that the condition is fulfilled for the relation R defined below.

Definition 6.1. *The domain of the relation R_{random} is $\{\eta : \eta$ is a code for a measure 1 set, say a tree $T_\eta \subseteq {}^{<\omega}2$ of positive measure$\}$. The range of R_{random}*

is ${}^{\omega}2$. *We set* $\eta R_{random}\nu$ *iff* $\nu \in A_\eta := \{\rho \in {}^{\omega}2 \, : \, \text{for some } \rho' \in T_\eta \text{ we have that } \rho =^* \rho'\}$.

Claim 6.2. *(1) Assume that*

$(\otimes_R)$

(a) R is a 2-place Borel relation on ${}^{\omega}2$, and

(b) for every $x_1, x_2 \in {}^{\omega}2$, if x_2 is not recursive, there is $x \in {}^{\omega}2$ such that

$$\otimes \qquad (\forall \nu)\Big(xR\nu \to (x_1 R\nu \wedge \neg(x_2 \leq_T \nu))\Big).$$

then the notions of strongly needed for R and weakly needed for R coincide and coincide with being recursive.

(2) The relation R_{random} satisfies the criterion $\otimes_R$ from Part (1).

Proof. (1) We have show that every weakly needed real is recursive. Then by "recursive $\to$ strongly needed $\to$ weakly needed $\to$ recursive" all three notions coincide.

Suppose that $x^* \in {}^{\omega}2$ is not recursive. We show that x^* is not weakly needed. Let Y be a strong R-cover. Let $Y^* = \{\nu \in Y \, : \, \neg x^* \leq_T \nu\}$. $Y^* \subseteq Y$, and hence $|Y^*| \leq |Y| = ||R||$. We show that Y^* is also an R-cover. Let $x_1 \in {}^{\omega}2$ be given. We take $x_2 = x^*$, and apply (b) of $\otimes_R$. So we get x as there. Since Y is an R-cover we find some $\nu \in Y$ such that $xR\nu$. Hence by $\otimes_R$ we have that $x_1 R\nu \wedge x_2 \not\leq_T \nu$. So $\nu \in Y^*$ R-covers x_1.

(2) Let x_1, x_2 be given. We take $N \prec (H(\beth_3), \in)$ such that $x_1, x_2 \in N$. Let $T = T_\eta$ be Amoeba-generic over N. Then $\eta = x$ is as claimed in (1)(b).

$\square$

Conclusion 6.3. *Strongly R_{random}-needed and weakly R_{random}-needed coincide and are just all the recursive reals.* $\square$

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